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Analysis Comprehensive Exam

9:00AM–11:30AM, June 19, 2018

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(1) Let $E \subset \mathbb{R}$ be uncountable. Prove that $\exists x \in \mathbb{R}$ such that $E \cap (-\infty, x)$ and $E \cap (x, \infty)$ are both uncountable.

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(2) Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers and let $f\colon [-1,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} a_n, & x = 1/n, \quad n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Find a necessary and sufficient condition on (a_n) for each item below so that f is

(a) continuous at 0,
(b) differentiable at 0,
(c) Riemann integrable on $[-1,1]$.

Note: For part (c) you cannot simply quote a theorem that says "a function with countably many discontinuities is Riemann integrable", because this is not a theorem from Rudin and was not proved in lectures.

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More space for Problem (2):

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(3) Let $f\colon [0,1] \to [0,1]$ be convex. Prove that the arclength of the graph of f is at most 3.

Recall that f is convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for any $x, y \in [0,1]$ and $\lambda \in [0,1]$.

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(4) Suppose that $f : [a, b] \to \mathbb{R}$ is a function. We say that f is:

- *absolutely continuous* if, for every $\epsilon > 0$, there is $\delta > 0$ such that, whenever (x_i, y_i), $i = 1, \ldots, k$, is a finite sequence of disjoint subintervals of $[a, b]$ with $\sum_{i=1}^{k}(y_i - x_i) < \delta$, we have $\sum_{i=1}^{k} |f(y_i) - f(x_i)| < \epsilon$.

- *of bounded variation* if there is $M > 0$ such that, whenever $P = \{a = x_0, x_1, \ldots, x_n = b\}$ is a partition of $[a, b]$, we have $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq M$.

(a) Suppose that f is C^1. Prove that f is absolutely continuous.

(b) Suppose that f is absolutely continuous. Prove that f is of bounded variation.

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More space for Problem (4):

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(5) Let $f : X \to Y$ be a continuous function where X and Y are metric spaces. Show that $f(K)$ is compact whenever $K \subset X$ is compact.

Note: This is a theorem from Rudin, but you must give a proof and cannot simply quote it.

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(6) In a metric space X, with metric d, let E be a nonempty subset of X. Define $f_E : X \to \mathbb{R}$ by

$$f_E(x) = \inf\{d(x, y) : y \in E\}$$

for each $x \in X$. Prove

(a) f_E is uniformly continuous on X.

(b) $\overline{E} = \{x \in X : f_E(x) = 0\}$.

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(7) Let $f : I \to \mathbb{R}$ be continuous where $I = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$. Show that

$$g(x_1, \ldots, x_{n-1}) = \int_{a_n}^{b_n} f(x_1, \ldots, x_{n-1}, x_n) dx_n$$

is continuous on $[a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \subset \mathbb{R}^{n-1}$.

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(8) Suppose that (X, d_X) and (Y, d_Y) are metric spaces. We make $X \times Y$ into a metric space by equipping it with the metric $d((x_1, y_1), (x_2, y_2)) := \max(d_X(x_1, x_2), d_Y(y_1, y_2))$. Show that $X \times Y$ is connected if and only if both X and Y are connected.

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(9) Let $\Sigma : [0,1]^2 \to \mathbb{R}^3$ be the 2-surface given by

$$\Sigma(\theta, \varphi) := (\sin \pi\theta \cos 2\pi\varphi, \sin \pi\theta \sin 2\pi\varphi, \cos \pi\theta).$$

Prove that $\partial\Sigma = 0$.

Note: This is an example in Rudin, but you cannot simply quote it.