Applied Mathematics Qualifying Exam

June 21, 2024

Time limit: 2.5 hours

Instructions: This exam has three parts A, B, and C, each of which contains three problems. Choose TWO problems from each of Parts A and C, and in Part B, you MUST do Problem 1 and then choose ONE of problems 2 and 3, for a total of SIX problems.

Part A

Choose any TWO of the following problems.

1. Find a leading order multiple scales expansion of the solution to the equation

$$\begin{aligned} y'' + y + \varepsilon\left(\tfrac{1}{3}(y')^3 - y'\right) &= 0, \qquad t > 0 \\ y(0) = 0, y'(0) &= 1, \end{aligned}$$

   where $\varepsilon > 0$ is a small parameter. *Hint*: you may use the fact that the solution to the ODE $2\frac{\mathrm{d}r}{\mathrm{d}\tau} = r - r^3$ satisfying $r(0) = r_0$ is given by

$$r(\tau) = \frac{r_0}{\sqrt{r_0^2 + (1 - r_0^2)e^{-\tau}}}$$

2. (a) Consider the vector field $\dot{u} = f(u)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^k for some $k \geq 1$. Define the notion of a Lyapunov function. Show that if the system admits a (strong) Lyapunov function, then there can be no periodic orbits.

   (b) Consider the vector field below

$$\begin{aligned} \dot{x} &= -y - 2x^3 \\ \dot{y} &= 2x + py - y^3 - 4x^2 y \end{aligned}$$

   where p is a parameter. When $p \leq 0$, by considering the function $V(x, y) = 2x^2 + y^2$, show that the system admits no periodic orbits. When $0 < p < 1$ show that there exists a periodic orbit inside the region bounded by the ellipse $V(x, y) = 1$. *Hint*: Use the Poincaré–Bendixon theorem.

3. Consider the equation below

$$\begin{aligned} \dot{x} &= px + x^2 - xy \\ \dot{y} &= x - y + 2x^2. \end{aligned}$$

(a) Compute the linearization of the system, and show that the origin undergoes a bifurcation at $p = 0$.

(b) For values of $p \approx 0$, find the flow on the associated center manifold, classify the type of bifurcation that occurs, and sketch the corresponding bifurcation diagram.

(c) For what values of $p \in \mathbb{R}$ is the origin locally asymptotically stable? Explain your answer.

Part B

You must complete problem 1, and then choose ONE of problems 2 or 3.

1. (**Mandatory**) Consider the following one step method

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right), \tag{1}$$

for $n = 1, 2, \ldots$, which approximates the solution of ordinary differential equations of the form

$$y' = f(x, y).$$

   (a) Determine the order of accuracy of the numerical method in Eq. (1).

   (b) State conditions for the one-step numerical method in Eq. (1) to be convergent.

   (c) Define what it means for a method to be A-stable.

   (d) Is the method in Eq. (1) A-stable?

2. Least squares.

   (a) Given a matrix $A \in \mathbb{R}^{m\times n}$ with rank equal to n. Show that the matrix $A^T A$ is symmetric, positive definite.

   (b) Suppose $b \in \mathbb{R}^m$. Show that $x = (A^T A)^{-1} A^T b$ minimizes $||Ax - b||_2$.

   (c) If the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}$, find the pseudoinverse $A^\dagger$.

3. Eigenvalues. Let $A = \begin{pmatrix} 20 & 0.1 & 0 \\ 0.1 & 20 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   (a) Define the power method for computing the largest eigenvalue of a symmetric matrix.

   (b) Explain why the power method for computing the largest eigenvalue of A converges slowly.

   (c) Choose a suitable shift σ so that the power method applied to $A - \sigma I$ converges to the largest eigenvalue of A the fastest.

Part C

Choose any TWO of the following problems.

1. Consider the admissible set

$$\mathcal{M} = \{x \in C^1([1,2]), x(1) = 1, x(2) = 2\},$$

and the problem

$$\min_{x\in\mathcal{M}} I(x) = \int_1^2 (\dot{x}(t) + t^2\dot{x}^2(t))\,\mathrm{d}t.$$

Verify $x^*(t) = 3 - 2/t$ is a weak minimum and also a strong minimum.

2. Let $L(t,x,v) = e^{-x}(1+v^2)^{1/2}$ be the Lagrangian.

a) Find Hamiltonian and solve the Hamilton system.

b) Write Hamilton-Jacobi equation and solve it.

3. Historically, the existence proof relied on Dirichlet's principle, which states: "If $I(\cdot)$ is bounded below, then there exists a minimizer u such that $I(u) = \inf_{v\in\mathcal{M}} I(v)$."

Here is a "proof"" of the Dirichlet's principle (provided by Riemann): "Choose a minimizing sequence $\{u_k\} \subset \mathcal{M}$ such that $I(u_k) \to \inf_{u\in\mathcal{M}} I(u)$. Since the sequence $\{u_k\}$ is bounded, there exists a convergent sub-sequence $u_{k_j} \to u_0$. This u_0 is then the desired solution, i.e., $I(u_0) = \inf_{u\in\mathcal{M}} I(u)$."

Find flaw in the above proof and give examples to support your finding.