Applied Mathematics Qualifying Exam

September 16, 2024

Time limit: 2.5 hours

Instructions: This exam has three parts A, B, and C, each of which contains three problems. Choose TWO problems from each of Parts A and C, and in Part B, you MUST do Problem 1 and then choose ONE of problems 2 and 3, for a total of SIX problems.

Part A

Choose any TWO of the following problems.

1. Consider the following boundary value problem for $y(x)$

$$\varepsilon y'' + y' - x^\beta y = 0, \quad 0 < x < 1$$
$$y(0) = y(1) = 1$$

   where $0 < \varepsilon \ll 1$ and $\beta > -1$. Find a leading order approximation to the solution using boundary layer theory.

2. Let $\mathcal{X} = C^0([-\varepsilon, \varepsilon], \mathbb{R}^n)$ be the space of continuous functions with the supremum norm $\|u\| = \sup_{t\in[-\varepsilon,\varepsilon]} |u(t)|$. Consider the map $T : \mathcal{X} \to \mathcal{X}$ defined by

$$(Tu)(t) = u_0 + \int_0^t f(u(s))\mathrm{d}s,$$

   where $u_0 \in \mathbb{R}^n$, and f is locally Lipschitz. That is, there exist $K, \delta > 0$ such that $|f(u) - f(v)| \leq K|u - v|$ for all $u, v \in B_\delta(u_0) = \{u \in \mathbb{R}^n : |u - u_0| < \delta\}$.

   (a) Show that T is a contraction mapping on the space $\mathcal{B} = \{u \in \mathcal{X} : \sup_{t\in[-\varepsilon,\varepsilon]} |u(t) - u_0| < \delta\}$ for an appropriate choice of $\varepsilon > 0$. What can be deduced about fixed points of the map T? (You may quote the contraction mapping theorem without proof).

   (b) What does part (a) imply about solutions to the initial value problem $\dot{u} = f(u), u(0) = u_0$?

   (c) Give an example of a *continuous* vector field $f : \mathbb{R} \to \mathbb{R}$ and $u_0 \in \mathbb{R}$ such that the initial value problem in part (b) has multiple solutions (you should find two solutions explicitly).

   (d) Now suppose that $f \in C^k(\mathbb{R}^n), k \geq 1$. Describe the smoothness (in t and u_0) of the solution $u(t; u_0)$ to the initial value problem $\dot{u} = f(u), u(0) = u_0$.

3. Consider the following ordinary differential equation

$$\dot{x} = x(1 - 3x^2 - 4y^2) - 2y$$
$$\dot{y} = y(1 - 4x^2 - 3y^2) + 2x$$

(a) Show that the system has at least one periodic orbit (use polar coordinates).

(b) Let $(x_*(t), y_*(t))$ denote the periodic orbit from part (a), with period T (you do not need to find x_*, y_* or T explicitly). Find the Floquet multipliers associated with the linearization about this orbit (your answer may include integral expressions involving x_*, y_* and/or T).

(c) Determine whether the periodic orbit is locally asymptotically stable.

Part B

You must complete problem 1, and then choose ONE of problems 2 or 3.

1. (**Mandatory**) Consider a system of two ODEs of the form:

$$\frac{dx}{dt} = f(x,y), \quad \frac{dy}{dt} = g(x,y).$$

Suppose that it is more computationally expensive to evaluate g than to evaluate f.

(a) Show that the multi-rate explicit Euler method defined by

$$\begin{aligned} x_{k+1/2} &= x_k + \tfrac{\Delta t}{2} f(x_k, y_k) \\ x_{k+1} &= x_{k+1/2} + \tfrac{\Delta t}{2} f(x_{k+1/2}, y_k) \\ y_{k+1} &= y_k + \Delta t g(x_k, y_k) \end{aligned}$$

is locally second order accurate, where Δt is the time step. Only consider the order at integer time subscripts.

(b) Apply the method from (a) to the system

$$\frac{dx}{dt} = -x + y, \quad \frac{dy}{dt} = -y.$$

Under what conditions on the time step Δt will the discrete solution remain stable, i.e., $\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = 0$ for any initial condition?

2. Let A be a $m \times n$ matrix and b be a $m \times 1$ vector. Consider the least squares problem: Find x that minimizes $\|Ax - b\|_2^2$.

(a) Give necessary and sufficient conditions for x to be a solution of the least squares problem.

(b) When is the solution unique?

(c) When is the corresponding residual vector $r = Ax - b$ unique?

(d) Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Find the least squares solution x that also minimizes $\|x\|_2^2$.

3. (a) Let A be a symmetric, $n \times n$ matrix. Show that the sum of the eigenvalues of A is equal to the sum of the diagonal elements of A. **Hint:** Compare $\det(\mathrm{A} - \lambda\mathrm{I})$ and $p(\lambda) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$.

   (b) Consider solving $Ax = b$, where A is $n \times n$ nonsingular. Derive the estimate for the relative error in the solution $\frac{\|\delta x\|}{\|x\|}$ if the right hand side b is perturbed by δb.

   (c) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1+\epsilon \end{pmatrix}$, $\epsilon > 0$. How small should ϵ be for you to call the matrix ill-conditioned?

Part C

Choose any TWO of the following problems.

1. Let a and b be two points on the unit sphere. Find the shortest path on the sphere connecting these two points.

   (a) Use the spherical coordinate to write a parametrization of the curve and then find the functional for the length of curves connecting a and b.

   (b) Solve the corresponding Euler-Lagrange equation.

   (c) Justify the solution is the global minimum.

2. Consider a frictionless harmonic oscillator consisting of a mass m on a spring of constant k. The potential energy is $V = \frac{1}{2}kx^2$ and the kinetic energy is $T = \frac{1}{2}m\dot{x}^2$.

   (a) Write out the Lagrangian and the Euler-Lagrange equation.

   (b) Write out the Hamiltonian and the Hamilton system.

   (c) Solve the Hamilton system for the position as a function of time.

3. Assume $F \in C^2$ and $|F''| \leq \lambda_1$, where λ_1 is the minimal eigenvalue of $-\Delta$ operator with homogeneous Dirichlet boundary condition on Ω. Consider

$$\min_{u \in H_0^1(\Omega)} I(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u) \right) \mathrm{d}x.$$

   Assume $u_0 \in H_0^1(\Omega)$ solves the Euler-Lagrange equation. Prove it is minimum.