Print Your I.D.Number: ________ ____________

Qualifying Examination, June 19, 2019

1:00 PM – 3: 30 PM, Room RH **306**

Table of your scores

Problem 1	———/ 10
Problem 2	———/ 10
Problem 3	———/ 10
Problem 4	———/ 10
Problem 5	———/ 10
Problem 6	———/ 10
Problem 7	———/ 10
Problem 8	———/ 10
Total	———/ 80

Notation:

$\mathbf{C}$ denotes the complex plane; $i = \sqrt{-1}$;

$D(z_0, r)$ denotes the open disc in $\mathbf{C}$ centered at z_0 and radius r;

$U = \{z = x + iy : y > 0\}$ is the upper half plane in $\mathbf{C}$.

1. Let $f(z)$ be analytic in the region $0 < |z| < 1$, which satisfies $\mathrm{Re} f(z) < 2$. Show that $z = 0$ is a removable singularity of f.

2. Let $u : \mathbf{C} \to \mathbf{R}$ be a nonconstant real harmonic function. Show that there exists a sequence of points $\{z_n\} \in \mathbf{C}$ such that $\lim_{n\to\infty} u(z_n) = -\infty$.

3. a) State the Schwarz-Pick Lemma.

b) Suppose $f : D(0,1) \to D(0,1)$ is a holomorphic mapping such that $f(0) = \frac{1}{6}$. Give an upper bound for $|f'(0)|$, and characterize the functions for which the upper bound is an equality.

4. Let

$$f(z) = \frac{z^3}{(z^2+1)\, e^{1/z}}$$

a)Find and classify all the singularities of $f(z)$ in the extended complex plane.
b) Evaluate

$$\oint_\Gamma f(z) dz,$$

where $\Gamma = \{z \in \mathbf{C} \mid |\mathrm{Re}\, \mathbf{z}| + |\mathrm{Im}\, \mathbf{z}| = \mathbf{3}\}$.
c) Does there is a holomorphic function F on $|z| > 3$ such that $F'(z) = f(z)$ on $|z| > 3$? (Justify your answer)

5. Assume that the power series

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

defines a holomorphic function for $|z| < 1$ and assume that

$$|a_1| > \sum_{j=2}^{\infty} j\,|a_j|\,.$$

Show that the function $f(z)$ is one-to-one in the open unit disk $D(0,1)$.

6. Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ denote an entire function satisfying the estimate

$$|f(z)| \leq 10e^{|z|}, \quad \text{for all } z \in \mathbf{C}.$$

Prove that the coefficients a_j satisfy

$$|a_j| \leq 10\big(e/j\big)^j, \quad \text{for all } \ j \in \mathbf{N}.$$

7. Construct a conformal map ϕ which maps D_1 onto D_2, where

$$D_1 = \{z \in \mathbf{C} : 0 < \mathrm{Re} z < 2\} \text{ and } D_2 = \{z \in \mathbf{C} : |z| > 1\}.$$

8. Let u be harmonic in $D(0,1) \setminus \{0\}$ satisfying

$$\lim_{z \to 0} \frac{u(z)}{\ln |z|} = 0.$$

Prove that u is harmonic on $D(0,1)$.