Complex Analysis Qualifying Exam

June 13, 2022

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Problem 1: Let $\varepsilon > 0$ and let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a polynomial with complex coefficients such that $|a_k - 1| \leq \varepsilon$ for all $k = 0, \ldots, n$. Show that for $\varepsilon > 0$ sufficiently small, $P(z)$ has n simple zeros in the annulus $\{z \in \mathbb{C} : \frac{1}{2} < |z| < 2\}$.

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Problem 2: Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Compute

$$\sup\{\mathrm{Im}\, f'(i/2) : \ f : H \to \mathbb{D} \text{ holomorphic}\}.$$

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Problem 3: Let f be holomorphic on $H = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and continuous on $\overline{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\}$. Suppose there exist some $C, \epsilon > 0$ such that

$$|f(z)| \leq \frac{C}{(|z|+1)^{\epsilon}}, \quad z \in \overline{H}.$$

Prove that

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt,$$

for all $z \in H$.

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Problem 4: Let f be a holomorphic function defined on a neighborhood of $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. Assume that whenever $|z| = 1$, one has $\operatorname{Re} f(z) = \operatorname{Im} f(z)$. Show that f must be a constant.

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Problem 5: Find all holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ such that for all $z \in \mathbb{C}$

$$f(z+1) = e^{2\pi} f(z) \quad \text{and} \quad f(z+i) = f(z).$$

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Problem 6: Let f be a holomorphic function on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and assume that $0 < |f(z)| \leq 1$ when $z \in \mathbb{D}$. Prove that for all $z \in \mathbb{D}$

$$|f(0)|^{\frac{1+|z|}{1-|z|}} \leq |f(z)| \leq |f(0)|^{\frac{1-|z|}{1+|z|}}.$$

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Problem 7: Let $\mathcal{E}_M$ be the set of functions holomorphic on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ that satisfy

$$\int_0^{2\pi} |f(e^{it})|^2 dt \leq M < \infty.$$

Show that $\mathcal{E}_M$ is a normal family.

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Problem 8: Determine all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$|f(z)| \leq e^{|z|},$$

for all large values of $|z|$, and

$$f(k^{\frac{1}{4}}) = k,$$

for all $k \in \mathbb{N}$.

Hint: Note that $f(z) = z^4$ satisfies the above conditions.