Complex Analysis

Qualifying Exam

Friday, September 18, 2020 — 4:00 pm -6:30 pm

Problem	1	2	3	4	5	6	7	8	Σ
Points									

Math Exam ID #:

Problem 1.

Prove that there exists no holomorphic function f such that

$$f(z) = u(x, y) + (2 + 3i)y^2, \quad z = x + iy,$$

where $u(x, y)$ is a real valued function.

Problem 2.

Let $f(z)$ be an entire function such that $|f(z)| \geq |z|^\alpha$ for all $|z| \leq 1$, where $\alpha = \frac{2019}{2020}$. Prove that $|f(0)| \geq 1$.

Problem 3.

Prove that for any $a \in \mathbb{C}$ and $n \geq 2$, the polynomial $z^n + az + a$ has at least one root in the disk $\{|z| \leq 2\}$.

Problem 4.

Evaluate
$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 - \pi^2)} dx$$

Problem 5.

Suppose the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ is equal to 2. Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} 2^{\sqrt{n}}(|a_{2n}| + |a_{2n+1}|)z^{3n}.$$

Problem 6.

Find explicitly a conformal mapping of the domain

$$\{|z| < 1, \ \ \mathrm{Re}\, z > 0, \ z \notin [0, 1/2]\}$$

onto the unit disc.

Problem 7.

Suppose $\mathcal{F}_i$, $i = 1, 2, 3$, is a family of holomorphic functions on the unit disc $\mathbb{D} = \{|z| < 1\}$ given by

$$\mathcal{F}_1 = \{f : \mathbb{D} \to \mathbb{C},\ f \text{ is a polynomial whose coefficients are bounded by } 1\},$$

$$\mathcal{F}_2 = \{f : \mathbb{D} \to \mathbb{C},\ f \text{ is a polynomial of degree at most } 2020\},$$

$$\mathcal{F}_3 = \mathcal{F}_1 \cap \mathcal{F}_2.$$

Which of the families $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ are normal? Explain your answer.

Problem 8.

Let $U \subseteq \mathbb{C}, U \neq \mathbb{C}$, be a simply connected open set. Suppose $f : U \to U$ is holomorphic (non necessarily one-to-one), and $f(z)$ is not identically equal to z. Prove or disprove each of the following statements:

1) There exists at most one fixed point of f in U.

2) There exists exactly one fixed point of f in U.