Comprehensive Exam in Real Analysis

Tuesday, June 19, 2021 — 4:00-6:30pm

Problem	1	2	3	4	5	6	7	8	9	Σ
Points										

Each problem is worth 10 points. No books, notes, or calculators are allowed.

Student's name:

Problem 1.

Let $E \subset \mathbb{R}$ be an uncountable set. Prove that the set of limit points of E is also uncountable.

Problem 2.

Consider a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ given by

$$a_n = \frac{1^{\sqrt{2}} + 2^{\sqrt{2}} + 3^{\sqrt{2}} + \cdots + n^{\sqrt{2}}}{n^{\sqrt{2}+1}}.$$

Does it converge? If yes, find $\lim_{n\to\infty} a_n$.

Problem 3.

Let K be a nonempty compact metric space with metric d, and suppose $f\colon K \to K$ obeys $d(f(x), f(y)) < d(x, y)$ for all distinct $x, y \in K$. Prove that there is a unique $x_0 \in K$ with $f(x_0) = x_0$.

Problem 4.

Let $f(x)$ be a continuous function on $[0, 1]$. Prove that

$$\lim_{h \to 0^+} \int_0^1 \frac{h}{h^2 + x^2} f(x)\, dx = \frac{\pi}{2} f(0).$$

Problem 5.

Let K be a compact metric space, and let $\{f_n\}_{n\in\mathbb{N}}$ be a uniformly bounded equicontinuous family of functions $K \to \mathbb{R}$. For each $n \in \mathbb{N}$, define $g_n\colon K \to \mathbb{R}$ by

$$g_n(x) = \max\{f_1(x), \ldots, f_n(x)\}.$$

Prove that the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly.

Problem 6.

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} 0, & \text{if } (x,y) = (0,0); \\ \frac{xy^3}{x^2+y^4}, & \text{otherwise.} \end{cases}$$

a) Show that f is continuous;

b) Show that partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at every point $(x,y) \in \mathbb{R}^2$;

c) Is $f : \mathbb{R}^2 \to \mathbb{R}$ differentiable? Explain your answer.

Problem 7.

Prove that the function

$$f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=\sin(\sqrt{x^2+|\sin y|})$$

is uniformly continuous on $\mathbb{R}^2$.

Problem 8.

Let $f(x)$ be a twice differentiable function satisfying $\lim_{x\to 0} \frac{f(x)}{x} = a$. Does the series $\sum_{n=1}^{\infty}(-1)^n f(\frac{1}{n})$ converges? Does it converge absolutely? How does the answer depend on a?

Problem 9.

Let $U \subset \mathbb{R}^2$ be a region bounded by parabolas $y = x^2$ and $x = y^2$. Evaluate

$$\int_{\partial U} (y + x^{2021})dx + (2023x - e^{2022y})dy,$$

where the boundary ∂U is oriented counterclockwise.