Comprehensive Exam in Real Analysis

Monday, September 13, 2021 — 9:30 am - 12 pm

Problem	1	2	3	4	5	6	7	8	9	Σ
Points										

Each problem is worth 10 points. No books, notes, or calculators are allowed.

Student's name:

Problem 1.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that f is surjective.

Problem 2.

Show that the sequence $\{a_n\}_{n=1}^{\infty}$ defined recursively by

$$a_1 > 0,\ a_n = \sqrt{5a_{n-1} - 6} \text{ for } n \geq 2,$$

converges, and find its limit.

Problem 3.

Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Also, assume that $f(a) = f(b)$, and $|f'(x)| \leq 1$ for all $x \in (a, b)$. Prove that

$$|f(x_1) - f(x_2)| \leq \frac{b-a}{2} \text{ for any } x_1, x_2 \in [a, b].$$

Problem 4.

a) TRUE or FALSE (prove or give a counterexample): If a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then it is continuous on $[a, b]$.

b) TRUE or FALSE (prove or give a counterexample): If a bounded function $f : [a, b] \to \mathbb{R}$ has uncountably many discontinuities, then it is not Riemann integrable on $[a, b]$.

Problem 5.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that $f^{-1}(E) \subset \mathbb{R}^2$ is bounded whenever $E \subset \mathbb{R}$ is bounded. Prove that f attains either a maximum or a minimum value.

Problem 6.

Let $f\colon [0,1] \times [0,1] \to \mathbb{R}$ be continuous, and let $g\colon [0,1] \to \mathbb{R}$ be defined by

$$g(x) = \max\{f(x,y) \colon y \in [0,1]\}.$$

Prove that g is continuous.

Problem 7.

Suppose M is a connected metric space which is also locally pathwise connected (i.e. every point $x \in M$ has an open neighborhood $U(x) \subset M$ that is pathwise connected). Does it imply that M is pathwise connected? Prove or give a counterexample.

Problem 8.

Let $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}$ denote the unit sphere in $\mathbb{R}^3$. Evaluate the surface integral over $\mathcal{S}$:

$$\iint_{\mathcal{S}} (3x^2 + 4y + 5z) dA$$

Problem 9.

Let

$$f(x) = \exp\left(\sin(\|x\|^{2021}) - \|x\|\right), \quad x \in \mathbb{R}^n,$$

where $\|x\|$ is the standard Euclidean norm of a vector $x \in \mathbb{R}^n$. Prove that f is uniformly continuous on $\mathbb{R}^n$.