ADVISORY EXAM IN ANALYSIS COMPREHENSION

Friday, September 13, 2024 — 1:00-3:30pm

Problem	1	2	3	4	5	6	7	8	9	Σ
Points										

Each problem is worth 10 points. No books, notes, or calculators are allowed.
Please do not use the back of the sheets in your solutions.

Student's code:

Student's code:

Problem 1.

Show that for every $N \in \mathbb{N}$

$$\sum_{k=1}^{N} \frac{1}{k} \geq \ln(N+1)$$

Student's code:

Student's code:

Problem 2.

Let f be a continuous function on $[-1, 1]$ such that $\int_{-1}^{1} f(x)x^{2n}dx = 0$ for all integers $n \geq 0$. Show that $f(0) = 0$.

Student's code:

Student's code:

Problem 3.

Let $K_1, K_2 \subset l^2(\mathbb{N})$ be two compact sets in the Hilbert space $l^2(\mathbb{N})$. Define

$$S = \left\{ v \in l^2(\mathbb{N}) \mid v = v_1 + v_2 \text{ for some } v_1 \in K_1, v_2 \in K_2 \right\}.$$

Prove that S is compact.

Student's code:

Student's code:

Problem 4.

Let $f : (0, +\infty) \to (0, +\infty)$ be twice differentiable with $f'' < 0$.

a) Prove that f is increasing;

b) Prove that f is uniformly continuous.

Student's code:

Student's code:

Problem 5.

Which of the following statements are true? Explain your answer.

a) If (M, d) is a metric space, and $f : M \to M$ is a map such that for any distinct $x, y \in M$ one has $d(f(x), f(y)) < d(x, y)$, then f has exactly one fixed point in M.

b) If (M, d) is a complete metric space, and $f : M \to M$ is a map such that $d(f(x), f(y)) < d(x, y)$ for any distinct $x, y \in M$, then f has exactly one fixed point in M.

c) If (M, d) is a bounded metric space, and $f : M \to M$ is a map such that $d(f(x), f(y)) < d(x, y)$ for any distinct $x, y \in M$, then f has exactly one fixed point in M.

d) If (M, d) is a compact metric space, and $f : M \to M$ is a map such that $d(f(x), f(y)) < d(x, y)$ for any distinct $x, y \in M$, then f has exactly one fixed point in M.

Student's code:

Student's code:

Problem 6.

Let the function $N(p,t)$, $N \in C^\infty(\mathbb{R} \times (0,\infty))$, satisfy the inequality

$$N(p+a, t+a^2) + N(p-a, t+a^2) \geq 2N(p,t)$$

for all $a, p \in \mathbb{R}$ and all $t > 0$. Show that then N must satisfy

$$\frac{1}{2}\frac{\partial^2}{\partial p^2}N(p,t) + \frac{\partial}{\partial t}N(p,t) \geq 0$$

for all $p \in \mathbb{R}$ and all $t > 0$.

Student's code:

Student's code:

Problem 7.

Let Q be a square in $\mathbb{R}^2$ with the vertices $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$. Evaluate

$$\int_C \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy,$$

where C is the boundary of Q oriented counterclockwise.

Student's code:

Student's code:

Problem 8.

Calculate the integral

$$\int_{\mathbb{R}^2} e^{-3x^2+2xy-3y^2}\,dxdy$$

Student's code:

Student's code:

Problem 9.

Give an example of a path connected bounded complete metric space that is not separable.

Student's code: