Print Your Exam Code: ____________

Exam Time: 3:00pm-5:30pm, September 12, 2022

Table of your scores

Problem 1 ________/ 10

Problem 2 ________/ 10

Problem 3 ________/ 10

Problem 4 ________/ 10

Problem 5 ________/ 10

Problem 6 ________/ 10

Problem 7 ________/ 10

Problem 8 ________/ 10

Problem 9 ________/ 10

Total ________/ 90

**Problem 1.**

(a) State the Axiom of Completeness for $\mathbb{R}$.

(b) Prove that a monotone increasing sequence of real numbers converges if and only if it is bounded above.

**Problem 2.** Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of continuous functions defined on $[a, b]$. Suppose $\sum_{n=1}^{\infty} f_n(x)$ converges pointwisely in $[a, b)$, but $\sum_{n=1}^{\infty} f_n(b)$ diverges. Show that $\sum_{n=1}^{\infty} f_n(x)$ does not converge uniformly on $[a, b)$.

**Problem 3.** Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Suppose that $\lim_{n\to\infty}(a_{2n} + 2a_n) = 0$. Prove that

$$\lim_{n\to\infty} a_n = 0.$$

**Problem 4.** Suppose a sequence of real valued functions $\{f_n\}$, $f_n : \mathbb{R} \to \mathbb{R}$, is given such that for all $n \in \mathbb{N}$ we have $f_n(0) = 0$, and for any $x, y \in \mathbb{R}$

$$|f_n(x) - f_n(y)| \leq \max\{|x|, |y|\} \cdot |x - y|.$$

Prove that the sequence $\{f_n\}$ has a subsequence that converges pointwise on $\mathbb{R}$.

**Problem 5.** Suppose $\{a_n\}$ and $\{b_n\}$ are monotone strictly decreasing sequences of positive real numbers converging to zero. Prove that if there exists a limit

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = A,$$

then we must have $\lim_{n\to\infty} \frac{a_n}{b_n} = A$.

**Problem 6.** Denote $U \subset \mathbb{R}^2$, $U = \{(x, y) : y > |x|\}$. Is it true that the map $f(x, y) = \left(\int_x^y e^{t^2}\,dt, \int_x^y e^{-t^2}\,dt\right)$ is a local diffeomorphism near every point $(x, y) \in U$? Explain your answer.

**Problem 7.** Let $\Sigma = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$. Compute the integral

$$\int_{\Sigma} x^3 \, dydz + y^3 \, dzdx + z^3 dxdy.$$

**Problem 8.** Prove or disprove that $f(x) = \ln(1+x^2+y^2)+\sin(x^2)$ is uniformly continuous on $\mathbb{R}^2$.

**Problem 9.** Let $(M, d)$ be a bounded metric space that has the following property: for any sequence $\{x_n\} \subset M$ with $d(x_n, x_{n+1}) \leq \frac{1}{n}$ there exists a convergent subsequence. Does it imply that $(M, d)$ is compact? Prove or give a counterexample.