1. Let $G$ be a group of size 42.

   (1) Prove that $G$ has a subgroup $H$ of order 6 and any two such subgroups are conjugate in $G$.

   (2) Deduce that $G = H \times N$, where $N$ is a normal subgroup of order 7.
2. Let $p$ be a prime. Show that for any Sylow $p$-subgroup $H \subset GL_n(\mathbb{F}_p)$ there exists a basis in the vector space $V = \mathbb{F}_p^n$ such that $H$ consists of $\mathbb{F}_p$-linear maps given, in that basis, by an upper-triangular matrix with 1 on the diagonal.
3. For a group $G$, let $G_1 := G$ and let $G_{n+1} := [G_n, G_n]$. We say $G$ is nilpotent if $G_N = 1$ for some $N$. Prove that if $G$ is a $p$-group, i.e. $|G| = p^n$ for some prime $p$, then $G$ is nilpotent.
4. Let $R \subseteq \mathbb{Q}$ be the subring in the field of rational numbers, given by the fractions $\frac{a}{b}$ with $a \in \mathbb{Z}$ and $b = 2^k3^l$ with $k, l \geq 0$. Describe the ideals of $R$. Is $R$ a PID?
5. Let $S = \{a + bi | a, b \in \mathbb{Z}\} \subset \mathbb{C}$ be the ring of Gaussian integers.

(1) Show that $S$ is a Euclidean Domain;
(2) Find a decomposition of $a = 11 \in S$ into a product of irreducibles in $S$;
(3) Find a decomposition of $b = 13 \in S$ into a product of irreducibles in $S$. 

6. Let $V$ be a nonzero finite-dimensional vector space over the complex numbers.

(1) If $S$ and $T$ are commuting linear operators on $V$, prove that each eigenspace of $S$ is mapped into itself by $T$.

(2) Let $A_1, \cdots, A_k$ be finitely many linear operators on $V$ that commute pairwise. Prove that they have a common eigenvector in $V$.

(3) If $V$ has dimension $n$, show that there exists a nested sequence of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V,$$

where each $V_j$ has dimension $j$ and is mapped into itself by each of the operators $A_1, \cdots, A_k$. 
7. Let $A$ be an $n \times n$ matrix with complex coefficients and assume that every eigenvalue $\lambda$ of $A$ satisfies $\text{Im}(\lambda) > 0$. Consider the $(2n) \times (2n)$ matrix

$$B = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

Find the invariant factors of $B$ in terms of invariant factors of $A$ and prove that $B$ is similar to a real valued matrix.
8. Find the Galois group of $x^6 - 2$ over $\mathbb{Q}$ and $\mathbb{F}_5$. 
9. Let $K/F$ be a finite Galois algebraic extension with no proper intermediate fields. Prove that $[K : F]$ is prime.
10. Let $Q$ denote the quaternion group, i.e.

$$Q = \{ \pm 1, \pm i, \pm j, \pm k \}$$

with $i^2 = j^2 = k^2 = ijk = -1$, $-1 \in Z(Q)$ and $(-1)^2 = 1$.

(1) Classify the conjugacy classes of $Q$
(2) Construct the character table of $Q$. 