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Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Each of the 10 questions is worth 10 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators. As usual, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers respectively, $\mathbb{F}_q$ denotes a finite field with q elements.

1. Let G be a group of size 42.

(1) Prove that G has a subgroup H of order 6 and any two such subgroups are conjugate in G.
(2) Deduce that $G = H \times N$, where N is a normal subgroup of order 7.

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2. Let p be a prime. Show that for any Sylow p-subgroup $H \subset GL_n(\mathbb{F}_p)$ there exists a basis in the vector space $V = \mathbb{F}_p^n$ such that H consists of $\mathbb{F}_p$-linear maps given, in that basis, by an upper-triangular matrix with 1 on the diagonal.

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3. For a group G, let $G_1 := G$ and let $G_{n+1} := [G, G_n]$. We say G is nilpotent if $G_N = 1$ for some N. Prove that if G is a p-group, i.e. $|G| = p^r$ for some prime p, then G is nilpotent.

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4. Let $R \subset \mathbb{Q}$ be the subring in the field of rational numbers, given by the fractions $\frac{a}{b}$ with $a \in \mathbb{Z}$ and $b = 2^k 3^l$ with $k, l \geq 0$. Describe the ideals of R. Is R a PID?

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5. Let $S = \{a + bi | a, b \in \mathbb{Z}\} \subset \mathbb{C}$ be the ring of Gaussian integers.

(1) Show that S is a Euclidean Domain;
(2) Find a decomposition of $a = 11 \in S$ into a product of irreducibles in S;
(3) Find a decomposition of $b = 13 \in S$ into a product of irreducibles in S.

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6. Let V be a nonzero finite-dimensional vector space over the complex numbers.

(1) If S and T are commuting linear operators on V, prove that each eigenspace of S is mapped into itself by T;

(2) Let $A_1, \cdots, A_k$ be finitely many linear operators on V that commute pairwise. Prove that they have a common eigenvector in V.

(3) If V has dimension n, show that there exists a nested sequence of subspaces

$$0 = V_0 \subset V_0 \subset V_1 \ldots \subset V_n = V,$$

where each V_j has dimension j and is mapped into itself by each of the operators $A_1, \cdots, A_k$.

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7. Let A be an $n \times n$ matrix with complex coefficients and assume that every eigenvalue λ of A satisfies $Im(\lambda) > 0$. Consider the $(2n) \times (2n)$ matrix

$$B = \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}$$

Find the invariant factors of B in terms of invariant factors of A and prove that B is similar to a real valued matrix.

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8. Find the Galois group of $x^6 - 2$ over $\mathbb{Q}$ and $\mathbb{F}_5$.

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9. Let K/F be a finite Galois algebraic extension with no proper intermediate fields. Prove that $[K : F]$ is prime.

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10. Let Q denote the quaternion group, i.e.

$$Q = \{\pm 1, \pm i, \pm j, \pm k\}$$

with $i^2 = j^2 = k^2 = ijk = -1$, $-1 \in Z(Q)$ and $(-1)^2 = 1$.

(1) Classify the conjugacy classes of Q.
(2) Construct the character table of Q.