Complex Analysis

Qualifying Exam

Thursday, September 20, 2018 — 1:00 pm -3:30 pm, Rowland Hall 114

Problem	1	2	3	4	5	6	7	8	9	Σ
Points										

Math Exam ID #:

1. Let f be an analytic function on $D(z_0, r) \setminus \{z_0\}$, where $r > 0$, such that $f(z) \neq 0$ for all $z \in D(z_0, r) \setminus \{z_0\}$. Consider the analytic function $g(z) = \frac{1}{f(z)}$ for $z \in D(z_0, r) \setminus \{z_0\}$. Prove that f has an essential singularity at z_0 if and only if g has an essential singularity at z_0.

2. Let f and g be entire functions. Suppose that

   (a) $g(z) \neq 0$ for all $z \in \mathbb{C}$,

   (b) $|f(z)| \leq \left|z^7 g(z)\right|$ for all $z \in \mathbb{C}$.

   Prove that there exists $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ such that
   $f(z) = \alpha z^7 g(z)$ for all $z \in \mathbb{C}$.

3. Let $\{f_n\}$ be a uniformly bounded sequence of analytic functions on the open unit disc $D(0,1)$. Suppose $\lim_{n\to\infty} f_n\left(\frac{1}{k}\right)$ exists for $k = 1, 2, \ldots$. Prove that there exists an analytic function f on $D(0,1)$ such that $f_n \to f$ uniformly on compact subsets of $D(0,1)$.

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4. Find the number of solutions with multiplicity of $e^z = 7z^9$ in the open unit disc around the origin.

5. Evaluate $\int_{\gamma} \frac{1+z}{1-\cos(z)} dz$, where

   (a) γ is the circle of radius 5 around 0, counterclockwise.

   (b) γ is the circle of radius 7 around 0, counterclockwise.

6. Find a surjective holomorphic map φ from the open unit disc $D = D(0,1)$ to the punctured disc $D^* = D \setminus \{0\}$, with $\varphi'(z) \neq 0$ for any $z \in D$.

7. Let $f\colon \{z \mid |z| > 0\} \to \mathbb{C}$ be analytic. Furthermore suppose that $\lim_{z\to\infty} f(z) = 0$. Show that for $|z| > 1$, one has

$$\frac{1}{2\pi i}\int_{\eta=1} \frac{f(\eta)}{\eta - z} = -f(z).$$

8. Suppose $f\colon D(0,1) \to D(0,1)$ is a holomorphic mapping such that $f(0) = \frac{1}{5}$. Give an upper bound for $|f'(0)|$, and characterize the functions for which the upper bound is an equality.

9. Let $u = \log(x^2 + y^2)$.

   (a) Find all harmonic conjugates of u in an open ball of radius 1 centered at 1.

   (b) Prove that there is no harmonic conjugate of u in $U \setminus \{0\}$, where U is any open set containing 0.