Real Analysis Qualifying Exam Fall 2018

September 21, 2018

Student's math exam ID#: ______________________________

INSTRUCTIONS: Do all work on the sheets provided. There is a blank page following each problem. **Please do not use the back of the sheets in your solutions**.

Problem	Point Value	Points Received
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	**60**	

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Problem 1 (10 points) Let $f_k : \mathbb{R}^n \to \mathbb{R}$, $k = 1, 2, \ldots,$ be a sequence of Lebesgue measurable functions. Prove that the set of points $x \in \mathbb{R}^n$ for which the sequence $(f_k(x))_{k=1}^{\infty}$ fails to converge as $k \to \infty$ is Lebesgue measurable.

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Problem 2 (10 points) Let $f : [0,1] \to \mathbb{R}$ be absolutely continuous. Suppose that $f(0) = 0$ and $f' \in L^p([0,1])$ for some $1 < p < \infty$. Show that for any $g \in L^{p'}([0,1])$, $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\|fg\|_{L^1([0,1])} \leq \left(\frac{1}{p}\right)^{1/p} \|f'\|_{L^p([0,1])} \|g\|_{L^{p'}([0,1])}.$$

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Problem 3 (10 points) Let $(f_k)_{k=1}^{\infty}$ be a sequence of functions in $L^p(\mathbb{R}^n)$, for some $1 < p < \infty$, such that $\|f_k\|_{L^p} \leq 2018$, $k = 1, 2, \dots$. Suppose that $f_k \to f$ almost everywhere on $\mathbb{R}^n$ as $k \to \infty$. Show that

(a) $f \in L^p(\mathbb{R}^n)$,

(b) $f_k \to f$ weakly as $k \to \infty$, i.e. for any $g \in L^{p'}(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\int f_k g dx \to \int f g dx.$$

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Problem 4 (10 points) Let $f \in L^2([0,1])$ be such that

$$\int_0^1 f(x)g(x)dx = 0$$

for all $g \in C([0,1])$ with the property that

$$\int_0^1 g(x)dx = \int_0^1 xg(x)dx = 0.$$

Prove that there are $a, b \in \mathbb{C}$ such that $f(x) = ax + b$ for almost all $x \in [0,1]$.

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Problem 5 (10 points) Let $f \in L^1(\mathbb{R})$. Show that the series

$$\sum_{n=1}^{\infty} f(nx)$$

converges absolutely for almost all $x \in \mathbb{R}$.

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Problem 6 (10 points) Let $F \subset \mathbb{R}^n$ be a closed set with $m(F^c) < \infty$, where m is the Lebesgue measure and $F^c = \mathbb{R}^n \setminus F$. Set

$$M(x) = \int_{\mathbb{R}^n} \frac{d(y, F)}{|x - y|^{n+1}} dy$$

where $d(y, F) = \inf_{x \in F} |y - x|$ is the Euclidean distance from y to F. Show that $M(x) < \infty$ for almost all $x \in F$ and $M(x) = \infty$ for all $x \in F^c$.

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