Real Analysis Qualifying Exam Fall 2019

September 20, 2019

Student's math exam ID#: ________________________________

INSTRUCTIONS: Do all work on the sheets provided. There is a blank page following each problem. **Please do not use the back of the sheets in your solutions**.

Problem	Point Value	Points Received
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	**60**	

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Problem 1 (10 points) Let $\Omega \subset \mathbb{R}^n$ be a compact set. Suppose we are taking the Lebesgue measure. Solve 3 of the following problems

(a) Prove or disprove that necessarily $L^4(\Omega) \subset L^3(\Omega)$.

(b) Prove or disprove that necessarily $L^3(\Omega) \subset L^4(\Omega)$.

(c) Prove or disprove that necessarily $L^4(\mathbb{R}^n) \subset L^3(\mathbb{R}^n)$.

(d) Prove or disprove that necessarily $L^3(\mathbb{R}^n) \subset L^4(\mathbb{R}^n)$.

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Problem 2 (10 points) Let E_n be a sequence of measurable sets in (X, μ) and

$$A = \{ x \in X \ : \ x \in E_n \text{ for infinitely many } n \}$$

(a) Assume $\mu(E_n) \to 0$ as $n \to \infty$. Prove or disprove that necessarily $\mu(A) = 0$.

(b) Suppose there exists an infinite subsequence $\{E_{n_i}\}$ such that $\mu(E_{n_i}) > \frac{1}{2}$. Prove or disprove that necessarily $\mu(A) > 0$. Discuss separately the cases of finite and infinite μ.

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Problem 3 (10 points) Let μ be a finite measure on $\mathbb{R}$ and define

$$f(x) = \int_{\mathbb{R}} \frac{\ln|x-t|}{|x-t|^{1/2}}\, d\mu(t), \quad x \in \mathbb{R}$$

Prove that $f(x)$ is finite λ-a.e. for $x \in \mathbb{R}$. Here λ denotes the Lebesgue measure on $\mathbb{R}$.

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Problem 4 (10 points) Let f be an absolutely continuous function on $\mathbb{R}$ such that $f \in L^1(\mathbb{R})$. Suppose that

$$\lim_{t \to 0^+} \int_{\mathbb{R}} \left| \frac{f(x+t) - f(x)}{t} \right| dx = 0.$$

Prove that $f \equiv 0$.

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Problem 5 (10 points) Let (X, μ) be a measure space and suppose that $f_n : X \to \mathbb{R}$ is a sequence of measurable functions such that

$$\sup_n \int_X |f_n|^2 \, d\mu < +\infty$$

and $\lim_{n\to\infty} \int_X (f_n \cdot g) \, d\mu$ exists and is finite for any $g \in L^2(X, \mu)$.

(a) Prove that there exists $f \in L^2(X, \mu)$ such that

$$\lim_{n\to\infty} \int_X (f_n \cdot g) \, d\mu = \int_X (f \cdot g) \, d\mu, \quad \textit{for } \forall g \in L^2(X, \mu).$$

(b) Prove that, for the f in part a), we have

$$\int_X |f|^2 \, d\mu \leq \liminf_{n\to\infty} \int_X |f_n|^2 \, d\mu.$$

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Problem 6 (10 points)

(a) Suppose $f \in L^2(\mathbb{R})$. Show that

$$\sum_{n=1}^{\infty} \frac{f(x+n)}{n^{3/4}} < \infty \text{ for a.e. } x \in \mathbb{R}.$$

(b) Give an example of $f \in L^4(\mathbb{R})$ such that

$$\sum_{n=1}^{\infty} \frac{f(x+n)}{n^{3/4}} = \infty \text{ for } \forall x \in \mathbb{R}.$$

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