Print Your Name: ______________ ______________
last first

Print Your I.D. Number: __________ ______________

Qualifying Examination, September 19, 2019

1:00 PM – 3: 30 PM, Room RH 306

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Notation:

$\mathbf{C}$ denotes the complex plane; $i = \sqrt{-1}$;
$\mathbf{R}$ is the set of all real numbers; $\mathbf{Z}$ is the set of all integers;
$D(z_0, r)$ denotes the open disc in $\mathbf{C}$ centered at z_0 and radius r;
$U = \{z = x + iy : y > 0\}$ is the upper half plane in $\mathbf{C}$.

1. Complete the following parts.

(a) Find all points where the function $f(z) = \overline{z}^2 + 3\overline{z}$ is analytic

(b) Let D be a domain in $\mathbf{C}$ and let f be C^1 function on D such that

$$\int_{\partial D(z_0,r)} f(z)dz = 0,$$

for any $z_0 \in D$ and $0 < r < \delta(z_0) =\text{dist}(z_0, \partial D)$. Prove f is holomorphic in D.

2. Let $f : D(0,1) \to D(0,1)$ be holomorphic such that $f(0) = 5^{-20}$. Give a sharp estimate for number of zeros of f on $\overline{D(0, \frac{1}{5})}$.

3. Given a series

$$\sum_{n=1}^{\infty} \left(\frac{2019+i}{2019-i}\right)^{n^2} \cdot \left(\frac{z-2019}{z+2019}\right)^{n}.$$

(i) Find all complex numbers z such that the series converges absolutely;
(ii) Find all complex numbers z such that the series convergence.

4. Prove that

$$\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{\pi^3}{8}.$$

5. Prove or disprove: there exist a sequence of holomorphic functions $\{f_n\}_{n=1}^{\infty}$ on $D(0,1)$ such that $f_n(z) \to |z|^2$ uniformly on a non-empty open subset of $D(0,1)$.

6. Suppose that f is a non-constant entire function which satisfying

$$|f(z)| \geq 1 \quad \text{when} \quad |z| \geq 10.$$

Prove that f is a polynomial.

7. Let $f : D(0,1) \to D(0,1)$ be a proper holomorphic map such that $f(z)$ is continuous on $\overline{D(0,1)}$. Prove f is a rational function.

8. (a) Prove

$$f(z) = \sum_{k=-\infty}^{\infty} \frac{1}{(k+z)^2} - \frac{\pi^2}{\sin^2(\pi z)}$$

is entire holomorphic.

(b) Prove

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+z)^2} = \frac{\pi^2}{\sin^2(\pi z)}, \quad z \notin \mathbf{Z}.$$

(Problem 8 continued)