ASSIGNED EXAM ID #: ____________________

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ALGEBRA QUALIFYING EXAM Fall **2019 QUESTIONS**

1. Do there exist groups G, H such that Q_8 is a semi-direct product of G and H?

2. Let G be a finite group of order $p^2 \cdot q$ where $p < q$ are primes. Prove that either G has a normal Sylow q-subgroup or G is isomorphic to A_4, the alternating group on 4 letters.

3. Let R be a ring with $1 \neq 0$. Let $N(R)$ be the set of all nilpotent elements of R. Prove the following.

(a) If R is commutative then $N(R)$ is an ideal of R and is equal to the intersection of all prime ideals of R.

(b) Give an example of a non-commutative ring R such that $N(R)$ is **not** an ideal of R.

4. Work with the ring $\mathbb{Z}[\sqrt{2}]$.

(a) Is $3 + \sqrt{2}$ a prime in $\mathbb{Z}[\sqrt{2}]$?

(b) Identify the quotient $\mathbb{Z}/(3 + \sqrt{2})$. Which ring it is?

You may use the fact that $\mathbb{Z}[\sqrt{2}]$ has a Euclidean norm.

5. Let R be a commutative ring with $1 \neq 0$. Consider two maximal ideals $I \neq J$ of R.

(a) Prove that every element of $R/(I \cap J)$ is either a unit or a zero divisor.

(b) Prove that $R/(I \cap J)$ always has a unit and at least two zero divisors.

(c) Prove that $R/(I \cap J)$ has infinitely many units if and only if it has infinitely many zero divisors.

Recall that a maximal ideal of R is, by definition, a proper ideal of R.

6. Assume $F \leq K \leq L$ are field extensions such that both K/F and L/K are algebraic. Prove that L/F is algebraic.

7. Calculate the Galois group of $x^4 - 3x^2 + 4$ over $\mathbb{Q}$.

8. Let $K \subseteq \mathbb{C}$ be a field extension of $\mathbb{Q}$ such that $\mathsf{Gal}(K/\mathbb{Q})$ is cyclic of order 4. Prove that $i \notin K$.

9. Let V be a finite dimensional vector space over $\mathbb{F}_2$ and let $T : V \to V$ be a linear transformation such that $T^5 = I$ but $T \neq I$.

(a) What are the possible degrees of the minimal polynomial of T?

(b) Assume there are non-simular linear transformation $S, T : V \to V$ such that $S^5 = I = T^5$ but $S \neq I \neq T$. What is the smallest possible dimension of V?

10. Prove that if A is a square matrix over $\mathbb{C}$ then A and its transpose A^T are conjugate over $\mathbb{C}$.