COMPLEX ANALYSIS

Qualifying Exam

Monday, January 4, 2021 — 4:00 pm - 6:30 pm

Problem	1	2	3	4	5	6	7	8	Σ
Points									

Math Exam ID #:

Problem 1.

For every $a \in \mathbb{R}$ find all solutions of the equation

$$|z|^2 + 2iz + 2a(1+i) = 0$$

Problem 2.

Evaluate $\int_\gamma f(z)dz$, if $f(z) = \frac{1}{(z-1)(iz-5)}$, and $\gamma(t) = 3e^{2\pi it} + e^{6\pi it}$, $t \in [0, 1]$.

Problem 3.

Let the sequence $a_0, a_2, a_3, \ldots$ be defined by the equation

$$1 + x^3 + x^6 + x^9 + \ldots = \sum_{n=0}^{\infty} a_n \left(x + \frac{1}{2}\right)^n, \quad (-1 < x < 0)$$

Find $\limsup_{n\to\infty}(|a_n|^{\frac{1}{n}})$.

Problem 4.

a) Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function such that both f^{2020} and f^{2021} are entire functions. Prove that f is an entire function.

b) Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function such that both f^{2020} and f^{2022} are entire functions. Does it imply that f is an entire function? Explain your answer.

Problem 5.

Let f be analytic in $\mathbb{C}\backslash\{0,2\}$, have a simple pole with residue -1 at $z=0$, and a simple pole with residue 1 at $z=2$. Prove that for any $K>0$ we have

$$\lim_{n\to\infty} K^n \left|\frac{f^{(n)}(1)}{n!}+1+(-1)^n\right| = 0$$

Problem 6.

How many roots does the function $f(z) = z^{2020} + z^{10} + 1$ have in the first quadrant?

Problem 7.

Find explicitly a conformal mapping that sends a domain

$$U = \{x + iy \mid 0 < x < 1,\ y > 0\}$$

on the complex plane to the unit disc.

Problem 8.

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, and set $g_a(z) = f\left(\left(\frac{z}{a}\right)^{\lfloor a \rfloor}\right)$, $a > 1$, where $\lfloor a \rfloor$ is the greatest integer less than or equal to a. Prove that $\{g_a\}_{a \in (1,+\infty)}$, $g_a : \mathbb{C} \to \mathbb{C}$, is a normal family.