Complex Analysis

Qualifying Exam

Wednesday, January 11, 2023 — 1:00 pm - 3:30 pm

Problem	1	2	3	4	5	6	7	8	Σ
Points									

Math Exam ID #:

Problem 1.

An *entire transcendental function* is an entire function which is not a polynomial. Prove that if f is an entire transcendental function and $K \subset \mathbb{C}$ is a compact set, then $f(\mathbb{C} \backslash K)$ is dense in $\mathbb{C}$.

Problem 2.

a) Give an example of a Laurent series centered at 1 that converges to the function $f(z) = \frac{z-2}{z(z-1)^2}$ in its domain of convergence.

b) Is an example that you gave unique? Explain your answer.

Problem 3.

Suppose that the power series $\sum_{n=0}^{\infty} c_n z^n$ has positive radius of convergence, and for some $\delta > 0$ the sum is real in the interval $(-\delta, \delta)$. Prove that all coefficients c_n are real.

Problem 4.

Suppose that $0 < |a| < 1$ and m is a positive integer. Prove that equation $(z-1)^m = a\mathrm{e}^{-z}$ has exactly m simple zeroes with positive real part, and that all of these zeroes are inside the disk D of radius 1 centered at 1, i.e. $D = \{z : |z-1| < 1\}$.

Problem 5.

Let $f(z) = \sum_{n\geq 0} a_n z^n$ have radius of convergence $R > 0$. Let $w_0 \in \partial D(0, R)$ be a point on the boundary of the disc of convergence, and suppose that for any $w \in \partial D(0, R)$, $w \neq w_0$, the function f can be analytically continued to an open set containing w.

a) Show that f cannot be analytically continued to any open set containing the point w_0.

b) Is it true that there exists sufficiently small $\varepsilon > 0$ such that f can be analytically continued to the open set $D(0, R + \varepsilon)\backslash\{w_0\}$? Explain your answer (prove or give a counterexample).

Problem 6.

Evaluate

$$\int_0^\infty \frac{\cos x}{4x^2 - \pi^2} dx$$

Problem 7.

Let $U \subset \mathbb{C}$ be given by

$$U = \{ z \in \mathbb{C} \mid \ |z| > 1 \ \text{ and } \ |z-1| < 2 \, \} \, .$$

Find a conformal mapping from U to the unit disc.

Problem 8.

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ be the upper half plane, and $f : \mathbb{H} \to \mathbb{C}$ be holomorphic and bounded. For a given $r > 0$ denote

$$\mathbb{H}_r = \{z \in \mathbb{C} \mid \text{Im}(z) > r\}.$$

Prove that for any $r > 0$ the restriction $f|_{\mathbb{H}_r} : \mathbb{H}_r \to \mathbb{C}$ is uniformly continuous.