COMPLEX ANALYSIS

Qualifying Exam

Wednesday, January 11, 2023 — 1:00 pm - 3:30 pm

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Math Exam ID #: 
Problem 1.

An *entire transcendental function* is an entire function which is not a polynomial. Prove that if $f$ is an entire transcendental function and $K \subset \mathbb{C}$ is a compact set, then $f(\mathbb{C} \setminus K)$ is dense in $\mathbb{C}$. 
Problem 2.

a) Give an example of a Laurent series centered at 1 that converges to the function \( f(z) = \frac{z^2 - 2}{z(z-1)^2} \) in its domain of convergence.

b) Is an example that you gave unique? Explain your answer.
Problem 3.

Suppose that the power series $\sum_{n=0}^{\infty} c_n z^n$ has positive radius of convergence, and for some $\delta > 0$ the sum is real in the interval $(-\delta, \delta)$. Prove that all coefficients $c_n$ are real.
Problem 4.

Suppose that $0 < |a| < 1$ and $m$ is a positive integer. Prove that equation \((z - 1)^m = ae^{-z}\) has exactly $m$ simple zeroes with positive real part, and that all of these zeroes are inside the disk $D$ of radius 1 centered at 1, i.e. $D = \{z : |z - 1| < 1\}$. 
Problem 5.

Let $f(z) = \sum_{n \geq 0} a_n z^n$ have radius of convergence $R > 0$. Let $w_0 \in \partial D(0, R)$ be a point on the boundary of the disc of convergence, and suppose that for any $w \in \partial D(0, R)$, $w \neq w_0$, the function $f$ can be analytically continued to an open set containing $w$.

a) Show that $f$ cannot be analytically continued to any open set containing the point $w_0$.

b) Is it true that there exists sufficiently small $\varepsilon > 0$ such that $f$ can be analytically continued to the open set $D(0, R + \varepsilon) \setminus \{w_0\}$? Explain your answer (prove or give a counterexample).
Problem 6.

Evaluate

$$\int_0^\infty \frac{\cos x}{4x^2 - \pi^2} \, dx$$
Problem 7.

Let $U \subset \mathbb{C}$ be given by

$$U = \{ z \in \mathbb{C} \mid |z| > 1 \text{ and } |z - 1| < 2 \}.$$

Find a conformal mapping from $U$ to the unit disc.
Problem 8.

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ be the upper half plane, and $f : \mathbb{H} \to \mathbb{C}$ be holomorphic and bounded. For a given $r > 0$ denote

$$\mathbb{H}_r = \{z \in \mathbb{C} \mid \text{Im}(z) > r\}.$$ 

Prove that for any $r > 0$ the restriction $f|_{\mathbb{H}_r} : \mathbb{H}_r \to \mathbb{C}$ is uniformly continuous.