Real Analysis Qualifying Exam Spring 2019

June 18, 2019

Student's math exam ID#: ______________________________

INSTRUCTIONS: Do all work on the sheets provided. There is a blank page following each problem. **Please do not use the back of the sheets in your solutions**.

Problem	Point Value	Points Received
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	**60**	

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Problem 1 (10 points) Let f_n be a sequence of functions in $L^\infty([0, 1])$ that converge to a function $f \in L^\infty([0, 1])$ in $L^2([0, 1])$. Prove, or disprove by example, that:

(a) f_n converge to f in $L^1([0, 1])$,

(b) f_n converge to f in $L^3([0, 1])$.

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Problem 2 (10 points) (a) Show that any sequence f_n of non-negative integrable functions on $[0, 1]$ with

$$\int_0^1 f_n^3 \, dx \leq \frac{1}{n^2}$$

must converge to zero almost everywhere.

(b) Is there a sequence g_n of non-negative integrable functions on $[0, 1]$ satisfying

$$\int_0^1 g_n^3 \, dx \to 0$$

which does not converge to zero almost everywhere? Explain.

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Problem 3 (10 points) Assume that ν and μ are two finite positive measures on a measure space (X, M). Prove that ν is absolutely continuous with respect to μ if and only if $\lim_{n\to\infty}(\nu - n\mu)_+ = 0$.

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Problem 4 (10 points) We say a function $f : [-1, 1] \to \mathbb{R}$ is convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $t \in [0, 1]$ and $x, y \in [-1, 1]$.

(a) Let $f(x)$ be a C^1 convex function on $[-1, 1]$. Show that f'' exists Lebesgue almost everywhere.

(b) Does there exist a C^1 convex function f on $[-1, 1]$ such that f'' equals zero almost everywhere, but f is not linear? Either construct or prove it is impossible.

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Problem 5 (10 points) Let $f \in L^1(\mathbb{R})$ and let g be the 1-periodic function on $\mathbb{R}$ with $g(x) = 1 - 2x$ for $0 \leq x < 1$. Show that

$$\lim_{t\to\infty} \int_{\mathbb{R}} f(x)g(x\sqrt{t})\, dx = 0.$$

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Problem 6 (10 points) Let f be a twice continuously differentiable function on $\mathbb{R}$ with compact support. Show that

$$\|f'\|_{L^2(\mathbb{R})} \leq \frac{1}{2} \left(\|f\|_{L^2(\mathbb{R})} + \|f''\|_{L^2(\mathbb{R})} \right).$$

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