Real Analysis Qualifying Examination, Tuesday, January 10, 2023  
12:00 PM – 2:30 PM, Room RH 306

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1. Consider a measure space \((X, \mathcal{A}, \mu)\) and a sequence of measurable sets \(E_n, n \in N\) such that

\[
\sum_{n=1}^{\infty} \mu(E_n) < \infty.
\]

Show that almost every \(x \in X\) is an element of at most finitely many \(E_n\)’s.
2. Let \((X, \mathcal{B}, \mu)\) be a \(\sigma\)-finite measure space and let \(f : X \to [0, \infty)\) be measurable. Let \(E := \{(x, y) \in X \times [0, \infty) : y \leq f(x)\}\). Assign the Lebesgue measure \(m\) on \([0, \infty)\). Prove that \(E\) is a measurable set on \(X \times [0, \infty)\) with respect to the product measure \(\mu \times m\) and that

\[
(\mu \times m)(E) = \int_X f d\mu.
\]
3. Suppose that $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ are measure spaces and $\Phi : X \to Y$ is a measurable map. Moreover, assume that for any measurable set $E \subset Y$, we have

$$\nu(E) = \mu(f^{-1}(E)).$$

Then for any measurable function $f : Y \to \mathbb{C}$, prove that $f \in L^1(\nu)$ if and only if $f \circ \Phi \in L^1(\mu)$, in which case,

$$\int_Y f d\nu = \int_X (f \circ \Phi) d\mu.$$
4. Let \( f_k \in L^1([0, 1]) \) for \( k \geq 1 \) (with respect to Lebesgue measure), and assume that \( \lim_{k \to \infty} \| f_k \|_{L^1([0, 1])} = 0 \).

   a) Show that a subsequence of \( \{ f_k \}_{k=1}^\infty \) tends to zero almost everywhere.

   b) Show by example that the sequence \( \{ f_k \}_{k=1}^\infty \) does not necessarily tend to zero almost everywhere.
5. Let $1 \leq p < q < \infty$ and $n \in \mathbb{N}$.

a) Show that the inclusions $L^p(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ are both false.

b) Show that, for any $r \in (p, q)$, we have $L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$, and furthermore that for $f \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ we have

$$\|f\|_r \leq \|f\|_p^{\alpha} \|f\|_q^{1-\alpha}, \quad \text{where} \quad \alpha = \frac{p(q - r)}{r(q - p)}.$$
6. Let $p \in (1, \infty)$. Suppose that $f_n \in L^p$ converges weakly to $f \in L^p$, that is, assume

$$\lim_{n \to \infty} \int_0^1 f_n g \, dx = \int_0^1 f g \, dx$$

for all $g \in L^q([0, 1])$, where $q = \frac{p}{p-1}$.

a) Show that $\|f\|_{L^p([0, 1])} \leq \liminf_{n \to \infty} \|f_n\|_{L^p([0, 1])}$.

b) Give an example where the inequality in part a) is strict.