Print Your Name: _______________ ________________
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Real Analysis Qualifying Examination, Tuesday, January 10, 2023

12:00 PM – 2:30 PM, Room RH 306

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1. Consider a measure space $(X, \mathcal{A}, \mu)$ and a sequence of measurable sets E_n, $n \in N$ such that

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty.$$

Show that almost every $x \in X$ is an element of at most finitely many E_n's.

2. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let $f : X \to [0, \infty)$ be measurable. Let $E := \{(x, y) \in X \times [0, \infty) \ : \ y \leq f(x)\}$. Assign the Lebesgue measure m on $[0, \infty)$. Prove that E is a measurable set on $X \times [0, \infty)$ with respect to the product measure $\mu \times m$ and that

$$(\mu \times m)(E) = \int_X f d\mu.$$

3. Suppose that $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ are measure spaces and $\Phi : X \to Y$ is a measurable map. Moreover, assume that for any measurable set $E \subset Y$, we have

$$\nu(E) = \mu(f^{-1}(E)).$$

Then for any measurable function $f : Y \to \mathbf{C}$, prove that $f \in L^1(\nu)$ if and only if $f \circ \Phi \in L^1(\mu)$, in which case,

$$\int_Y f d\nu = \int_X (f \circ \Phi) d\mu.$$

4. Let $f_k \in L^1([0, 1])$ for $k \geq 1$ (with respect to Lebesgue measure), and assume that $\lim_{k\to\infty} \|f_k\|_{L^1([0,1])} = 0$.

a) Show that a subsequence of $\{f_k\}_{k=1}^{\infty}$ tends to zero almost everywhere.

b) Show by example that the sequence $\{f_k\}_{k=1}^{\infty}$ does not necessarily tend to zero almost everywhere.

5. Let $1 \leq p < q < \infty$ and $n \in \mathbb{N}$.

a) Show that the inclusions $L^p(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ are both false.

b) Show that, for any $r \in (p, q)$, we have $L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$, and furthermore that for $f \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ we have

$$\|f\|_r \leq \|f\|_p^{\alpha} \|f\|_q^{1-\alpha}, \quad \text{where}\, \alpha = \frac{p(q-r)}{r(q-p)}.$$

6. Let $p \in (1, \infty)$. Suppose that $f_n \in L^p$ converges weakly to $f \in L^p$, that is, assume

$$\lim_{n\to\infty} \int_0^1 f_n g \, dx = \int_0^1 f g \, dx$$

for all $g \in L^q([0, 1])$, where $q = \frac{p}{p-1}$.

a) Show that $\|f\|_{L^p([0,1])} \leq \liminf_{n\to\infty} \|f_n\|_{L^p([0,1])}$.

b) Give an example where the inequality in part *a*) is strict.