Real Analysis Qualifying Exam

June 18, 2024

Student's math exam ID#: ______________________________

INSTRUCTIONS: Do all work on the sheets provided. There is a blank page following each problem. **Please do not use the back of the sheets in your solutions**.

Problem	Point Value	Points Received
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	**60**	

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Problem 1 (10 points)

Assume that $f \in L^2(\mathbb{R})$. Let $F(x) = \int_0^x f(t)\, dt$. Show that

$$\lim_{x \to \infty} \frac{F(x)}{\sqrt{x}} = 0.$$

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Problem 2 (10 points)

Suppose that $f(x)$ and $\{f_n(x)\}_{n\geq 1}$ are non-negative integrable functions on $\mathbb{R}$. Assume further that

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{a.e.} \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,dx = \int_{\mathbb{R}} f(x)\,dx.$$

Prove that for any measurable set $E \subset \mathbb{R}$,

$$\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E f(x)\,dx.$$

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Problem 3 (10 points)

Let $f : [0, 1] \to \mathbb{R}$ be non-decreasing. Recalling that this assumption implies that $f'(x)$ exists for almost all $x \in [0, 1]$ (with respect to Lebesgue measure dx), prove that $\int_0^1 f'(x)dx \leq f(1) - f(0)$.

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Problem 4 (10 points)

Suppose that $(X, \mathcal{B}, \mu)$ is a measure space and $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are integrable. For each $t \in \mathbb{R}$, set $A_t := \{x \in X \ : \ f(x) > t\}$ and $B_t := \{x \in X \ : \ g(x) > t\}$.

Part a: Assume that $(X, \mathcal{B}, \mu)$ is σ-finite. Prove that

$$\int_X |f - g| d\mu = \int_{-\infty}^{\infty} \mu(A_t \triangle B_t) dt.$$

Here, for subsets C and D of a set Y, $C \triangle D := (C \setminus D) \cup (D \setminus C)$ denotes their symmetric difference.

Part b: Show that the conclusion of part (a) holds even if $(X, \mathcal{B}, \mu)$ is not σ-finite.

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Problem 5 (10 points)

For $x \in \mathbb{R}$ and $t > 0$, let $\rho(x) := \max\{1 - |x|, 0\}$ and $\rho_t(x) := t\rho(tx)$.

Part a: Let u be a continuous function on $\mathbb{R}$ that vanishes outside of a compact set. Prove that the functions $u_t(x) := \int_{\mathbb{R}} \rho_t(x-y)u(y)\,dy$ converge uniformly to u on $\mathbb{R}$ as $t \to \infty$.

Part b: Let $u \in L^p(\mathbb{R})$ for some $p \in [1, \infty)$. Prove that the functions $u_t(x) := \int_{\mathbb{R}} \rho_t(x-y)u(y)\,dy$ converge to u in $L^p(\mathbb{R})$ as $t \to \infty$.

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Problem 6 (10 points)

Suppose $E \subset \mathbb{R}$ is a (Lebesgue) measurable subset. For all $x \in \mathbb{R}$, define $d(x, E) = \inf_{y \in E} |x - y|$. Prove that for a.e. $x \in E$,

$$\lim_{y \to 0} \frac{d(x + y, E)}{|y|} = 0.$$

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