Real Analysis Qualifying Exam
January 2021

There are 6 problems on the exam. Each problem is worth 10 points. Do as many as you can.
Show your work and justify your answers.

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Problem 1: A.) Give an example of a sequence $\{f_n\}_n$ of Lebesgue measurable functions on $[0, 1]$ that does not converge to zero almost everywhere, but nonetheless

$$\int_0^1 f_n^{10} dx \to 0$$

B.) Is there such an example if one assumes that

$$\int_0^1 f_n^{10} dx \leq \frac{1}{n^{10}}?$$

Give an example or prove there isn't one.

Problem 2: A.) Give an example of a measure μ on the unit interval that differs from Lebesgue measure such that:

(i) $\mu([0,1]) = 1$,

(ii) Every Borel set is μ-measurable,

(iii) If $x \in [0,1]$, then $\mu(\{x\}) = 0$.

(iv) If $(a,b) \subseteq [0,1]$ and $a \neq b$ then $\mu((a,b)) \neq 0$.

B.) Show that for all measures μ on the unit interval satisfying properties (i-iv) there is a measure ν on the unit square $[0,1] \times [0,1]$ such that

(a) $\nu([0,1] \times [0,1]) = 1$,

(b) Every Borel set is ν-measurable

(c) If $(x,y) \in [0,1] \times [0,1]$ then $\nu(\{(x,y)\}) = 0$

(d) For $A \subseteq [0,1]$

$$\nu(A \times [0,1]) = \mu(A).$$

Define the measure and prove it has properties (a)-(d).

Problem 3: Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be the measure spaces given by:

- $X = Y = [0, 1]$.
- $\mathcal{A} = \mathcal{B} = \mathcal{B}_{[0,1]}$, the Borel σ-algebra of $[0, 1]$.
- $\mu = \mu_L$ (Lebesgue measure) and ν is the counting measure.

Consider the product measurable space $(X \times Y, \sigma(\mathcal{A} \times \mathcal{B}))$ and a subset of it defined by $E = \{(x, y) \in X \times Y;\ x = y\}$. Then, show that

1. $E \in \sigma(\mathcal{A} \times \mathcal{B})$.
2. $\int_X \left\{ \int_Y \mathbf{1}_E \, d\nu \right\} \, d\mu \neq \int_Y \left\{ \int_X \mathbf{1}_E \, d\mu \right\} \, d\nu$.
3. Why is Tonelli's Theorem not applicable?

Problem 4: Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces, and consider the product measure space $(X \times Y, \sigma(\mathcal{A} \times \mathcal{B}), \mu \times \nu)$. Let f be a real-valued, $\mathcal{A}$-measurable and μ-integrable function on X and g a real-valued, $\mathcal{B}$-measurable and ν-integrable function on Y. Consider the real-valued function on $X \times Y$ given by $h(x, y) = f(x)g(y)$. Show that h is a $\sigma(\mathcal{A}\times\mathcal{B})$-measurable and $\mu\times\nu$-integrable function on $X\times Y$, and

$$\int_{X\times Y} h \, \mathrm{d}(\mu \times \nu) = \left\{ \int_X f \, \mathrm{d}\mu \right\} \left\{ \int_Y g \, \mathrm{d}\nu \right\}.$$

Problem 5: $g(x) : \mathbb{R} \to \mathbb{R}$ is a measurable function and O is non-empty open subset of $\mathbb{R}$. Let $h(t) = t \sin t$. Define

$$f(x) = \inf\{t \geq 0 |\ h(t)g(x) \in O\}.$$

Show that $f(x)$ is a real-valued measurable function.

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Problem 6: $\{f_n\}_{n\geq 1}$ is a sequence of real valued functions in $L^2([0,1])$. Suppose that for any $g \in L^2([0,1])$,

$$\sup_{n\geq 1} \int_{[0,1]} f_n g \, dx < +\infty,$$

Prove that

$$\sup_{n\geq 1} ||f_n||_{L^2([0,1])} < +\infty.$$

(You may NOT simply quote the Banach-Steinhaus Theorem or the Uniform Boundedness Principle.)