Qualifying Examination

Math Exam ID Number: ____________________

The exam consists of six problem. Solve as many as you can by giving complete arguments and providing explicit justification. Show all your work. Each problem is worth 5 points.

Problem	**Score**
1	
2	
3	
4	
5	
6	
Total	

Problem 1

Let $f_n : [-1, 1] \to \mathbb{R}$ converge pointwise to $f : [-1, 1] \to \mathbb{R}$. Show that

$$TV(f) \leq \liminf_{n\to\infty} TV(f_n).$$

Before embarking on the proof, state the definition of TV (total variation).

Problem 2

Let μ and ν be two positive measures on a measurable space $(X, \mathcal{M})$. Suppose that, for every $\varepsilon > 0$, there exists $E \in \mathcal{M}$ such that $\mu(E) < \varepsilon$ and $\nu(E^{\mathsf{c}}) < \varepsilon$. Show that $\mu \perp \nu$.

Problem 3

Let μ be a probability measure, i.e. a measure on X with $\mu(X) = 1$. Show that, given measurable $E_1, \ldots, E_n$, it holds that

$$\sum_{j=1}^{n} \mu(E_j) > n - 1 \Longrightarrow \mu\big(\bigcap_{j=1}^{n} E_j\big) > 0.$$

Problem 4

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f \in L^1(X, \mathcal{M}, \mu)$. Let $(E_n)_{n \in \mathbb{N}}$ be a sequence of $\mathcal{M}$-measurable sets such that $\lim_{n\to\infty} \mu(E_n) = 0$. Prove that

$$\lim_{n\to\infty} \int_{E_n} f \, d\mu = 0.$$

Problem 5

Suppose that $E \subset \mathbb{R}$ is a measurable set of positive measure and $D = \left\{ q_k \,\middle|\, k \geq 1 \right\} \subset \mathbb{R}$ be a dense set. Prove that

$$S = \mathbb{R} \backslash \bigcup_{k \geq 1} (q_k + E)$$

is a set of measure zero.

Problem 6

Suppose that $f \in L^1\big((0, +\infty)\big)$ and prove that

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n x f(x)\, dx = 0.$$