Algebra Qualifying Exam June 2020

1. Consider the cube C in the Euclidean vector space $\mathbb{R}^3$ as the set of points (x, y, z) for which the absolute value of each of the coordinates is less than or equal to 1. Let G be the group of those rotations in $\mathbb{R}^3$ preserving the origin which send C to itself. Show that G is isomorphic to the symmetric group S_4.

2. Let $p > 2$ be a prime. Prove that if G is a group of order $p^{p+3}(p^2 + p + 1)$ then G is not simple.

3. Determine whether the quaternion group Q_8 can be represented as a quotient of S_4. Justify your answer.

4. Show that $D = \mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain, with respect to the usual complex norm. Find an irreducible element of D which is not in $\mathbb{Z}$. Justify your answer.

5. Let R be a commutative ring with 1. Recall that an element r of R is idempotent iff $r^2 = r$.

Assume R is finite and strictly more than $\frac{2}{3}$ elements of R are idempotent. Let A be the set of all idempotent elements of R and $Z = A \cap (A - 1)$.

(a) Prove that $2a = 0$ whenever $a \in Z$.
(b) Prove that $(Z, +)$ is a subgroup of $(R, +)$ of index at most 2.
(c) Prove that $(A, +)$ is a subgroup of $(R, +)$ and conclude that every element of R is idempotent.

6. We say that a finitely generated R-module M is *invertible* iff there exist a finitely generated R-module N such that $M \otimes_R N \cong R$.

(a) Find all invertible $\mathbb{Z}$-modules.
(b) Prove that if M_1 and M_2 are both invertible then so is $M_1 \otimes_R M_2$.
(c) Prove that every invertible R-module is projective.

7. Consider the polynomial $g(x) = x^{20} + x^{10} + 1$.

(a) Find the splitting field of $g(x)$ if we consider $g(x)$ as a polynomial in $\mathbb{F}_5[x]$.
(a) Find the splitting field of $g(x)$ if we consider $g(x)$ as a polynomial in $\mathbb{F}_7[x]$.

8. Let $f(x) \in \mathbb{Q}[x]$ be a monic cubic polynomial with distinct roots r, s, t. Let $g(x)$ be the monic cubic polynomial with roots $r^2 + s + t$, $s^2 + t + r$ and $t^2 + r + s$.

(a) Prove that $g(x) \in \mathbb{Q}[x]$, that is, the coefficients of $g(x)$ are in $\mathbb{Q}$.
(b) Prove that if $\mathsf{Gal}(f) \cong S_3$ then $\mathsf{Gal}(g) \cong S_3$.

9. Let p be a prime, L be a finite field with p^2 elements and K be a finite field with p elements. Consider the map $F : L \to L$ taking x to its p-th power x^p. If L is viewed as a vector space over K, show that F is K-linear and find its characteristic and minimal polynomials.

10. Prove that for every $n \geq 3$ there exists an $n \times n$ nonsingular matrix $A \neq \pm I$ over $\mathbb{F}_3$ such that $I + A^2$ is its inverse.