ALGEBRA QUALIFYING EXAM JANUARY 6, 2021

1. Assume G is a finite group and H is a normal subgroup of G. Recall that $n_p(G)$ is the number of Sylow p-subgroups of G. Prove that $n_p(G/H) \leq n_p(G)$.

2. Decide: Is it possible for the symmetric group S_5 to act transitively on a set of cardinality 14? Provide a proof to support your claim.

3. Let R be a principal ideal domain. Suppose that $f, g, h \in R$ are such that $f = gh$ while g and h are relatively prime. Prove that

$$R/(f) \cong R/(g) \times R/(h).$$

4. Consider an integral domain R and its corresponding field of fractions F. Assume $p(x) \in R[x]$ is a monic pronominal and that it is possible to write $p(x)$ as a product

$$p(x) = q(x)r(x)$$

where $q(x), r(x) \in F[x]$ are monic polynomials of degree smaller than $\deg(p(x))$ and at least one of $q(x), r(x)$ is not in $R[x]$. Prove that R is not a unique factorization domain.

5. Let R be a commutative ring and let $e \in R$. For an R-module M, denote $eM = \{em : m \in M\}$, which is a submodule of M. If

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

is an exact sequence of R-modules and $e = e^2$, prove that

$$0 \to eL \xrightarrow{\alpha} eM \xrightarrow{\beta} eN \to 0$$

is also an exact sequence.

6. Consider an integral domain R and a principal ideal I of R, which is viewed as an R-module. Let M be the R-module $I \otimes_R I$. Prove that the only torsion element of M is zero.

7. Recall that $\mathbb{F}_q$ denotes the finite field with q elements. Find a polynomial $p(x) \in \mathbb{F}_2(x)$ such that $\mathbb{F}_2[x]/(p(x)) \cong \mathbb{F}_8$. Prove that your polynomial yields the desired isomorphism.

8. Consider a subfield F of the field of real numbers $\mathbb{R}$. Let $a \in F$ and $K = F(\sqrt[n]{a})$ where $\sqrt[n]{a} \in \mathbb{R}$ is an n-th root of a in $\mathbb{R}$ and n is odd.

Assume L is a Galois extension of F such that $L \subseteq K$. Prove that $L = F$.

Remark. The following theorems from the course may be useful.

A. Assume F is a field of characteristic 0 which contains all n-th roots of unity. Then the following holds: If $a \in F$ and b is an n-th root of a then the extension $F(b)/F$ is cyclic of order dividing n.

B. Assume K/F is a Galois extension of fields and F'/F is any finite extension of fields. Then KF'/F' is a Galois extension and

$$\mathsf{Gal}(KF'/F') \simeq \mathsf{Gal}(K/K \cap F')$$

.

9. Assume $A \in M_n(\mathbb{C})$ is a matrix over complex numbers such that all eigenvalues of A are non-zero. Prove that A has a square root in $M_n(\mathbb{C})$, that is, there is a matrix $B \in M_n(\mathbb{C})$ such that $A = B^2$.

Remark. It may be helpful to examine the Jordan form of the square of a Jordan cell.

10. Find a non-singular matrix $A \in M_n(\mathbb{F}_5)$ of smallest possible dimension n such that $A^2 + 2I$ is its own inverse and A is not a scalar multiple of I.