ALGEBRA QUALIFYING EXAM — JUNE 2022

UNIVERSITY OF CALIFORNIA, IRVINE

(1) If H is a subgroup of a finite index in G, show that H contains a subgroup N which is normal and of finite index in H.

(2) Prove that there is no simple group of order 351.

(3) If M, N are normal subgroups of G, show that $G/(M \cap N)$ is isomorphic to a subgroup of $(G/M) \times (G/N)$.

(4) Let F be a field. For any nonzero ideal I of $F[x]$, prove that there is an isomorphism of rings $F[x]/I \cong R_1 \times \cdots \times R_n$, where each R_i has the property that its ideals all lie in a finite chain $(0) = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{d_i} = R_i$.

(5) Assume the following rings are commutative with multiplicative identity. For each of the following, either give an example (with explanation) of such an ideal or show why it does not exist:
(a) A prime ideal in a finite ring that is not maximal.
(b) A prime ideal in an integral domain that is nonzero but not maximal.

(6) Let A, B be two finite abelian groups of orders n, m, respectively and assume that $gcd(n, m) \neq 1$. Show that $A \otimes_{\mathbb{Z}} B \neq 0$.

(7) Classify all modules over $\mathbb{Z}[i]$ that have 16 elements. (HINT: they are also modules over $\mathbb{Z}$.)

(8) Assume that an irreducible degree 7 polynomial over $\mathbb{Q}$ has a cyclic Galois group. Show that the order of this group must be 7.

(9) Let p be a prime such that $p \equiv 1 \pmod N$ for a positive integer $N \geq 2$. Suppose that $x^N - a$ is an irreducible polynomial over $\mathbb{F}_p$. Show that its splitting field has degree N over $\mathbb{F}_p$.

(10) Suppose that K/F is a field extension with $[K\colon F] = 5$. If $\alpha \in K$ is a root of a quadratic polynomial in $F[x]$, show that $\alpha \in F$.