ALGEBRA QUALIFYING EXAM SEPTEMBER 15, 2020

1. Consider $n \geq 2$. Prove that there are permutations $\sigma, \tau \in S_{2n}$ both of order 2 such that $\sigma \circ \tau$ has order n.

2. Give an example of a semi-direct product of two abelian groups which is not abelian. Justify your example by an explanation why it works.

3. Let $\mathbb{Q}(x)$ be the field of fractions of the integral domain $\mathbb{Q}[x]$, which is called the *field of rational functions*. For the subring

$$A = \left\{ \frac{f(x)}{g(x)} \in \mathbb{Q}(x) : g(0) \neq 0 \right\}$$

of $\mathbb{Q}(x)$, prove the following:

(a) A is a principal ideal domain.
(b) A has a unique irreducible element up to associates.

4. Consider the ideal I of the polynomial ring $\mathbb{Z}[x]$ which is generated by a prime number p and a non-constant polynomial $f(x) \in \mathbb{Z}[x]$. Prove that I is maximal if and only if $f(x)$ is irreducible modulo p.

5. Suppose that K is a field of characteristic 5. For which values of $n \geq 1$ is the polynomial $f(x) = x^n - x$ separable?

6. Let q be a prime power. Consider the finite field $\mathbb{F}_q$ as an abelian group under addition. For which q is this group cyclic?

7. Let $E < F$ be a field extension of degree 5 and K the smallest subfield in the algebraic closure of E, such that K is Galois over E and contains F. Show that the degree of K over E is at most 120.

8. Give an example of an injective map of abelian groups $M_1 \to M_2$, and an abelian group N, such that $M_1 \otimes_{\mathbb{Z}} N \to M_2 \otimes_{\mathbb{Z}} N$ is not injective. (Here $\otimes_{\mathbb{Z}}$ is the tensor product over the ring $\mathbb{Z}$ of integers.) Justify your example by an explanation why it works.

9. For a matrix $A \in \mathbb{M}_n(\mathbb{R})$, prove that the following are equivalent:

(a) the only eigenvalue of A is $\lambda = 0$;
(b) there exists $m \geq 1$ such that A^m is the zero matrix;
(c) A^n is the zero matrix.

10. Suppose that $T : V \to V$ is a linear operator on a finite dimensional vector space V over the field $\mathbb{Q}$ of rational numbers, and that T has characteristic polynomial which is irreducible over $\mathbb{Q}$. Show that the matrix of T (in any basis of V) can be diagonalized **over the field $\mathbb{C}$ of complex numbers**.