Qualifying Examination

Math Exam ID Number: ____________________

The exam consists of six problem. Solve as many as you can by giving complete arguments and providing explicit justification. Show all your work. Each problem is worth 5 points.

Problem	**Score**
1	
2	
3	
4	
5	
6	
Total	

Problem 1

In a topological space X, a subset $E \subset X$ is called a G_δ set if it can be written as the countable intersection of open sets. Consider a map $f : X \to Y$ between metric spaces X and Y. Prove that

$$C(f) = \{x \in X \mid f \text{ is continuous at } x\}$$

is a G_δ subset of X.

Problem 2

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set of finite Lebesgue measure. Suppose further that $f : E \to \mathbb{R}$ is a Lebesgue measurable function satisfying the following condition:

$$\forall \varepsilon > 0 \, \exists \delta > 0 \text{ s.t } \int_F |f| \, dm \leq \varepsilon \text{ whenever } E \supset F \in \mathcal{L} \text{ with } m(F) \leq \delta.$$

Prove that $f \in \mathrm{L}^1(E, m)$.

Problem 3

Suppose that $(X, \mathcal{B}, \mu)$ is a finite measure space. Define a pseudometric d on $\mathcal{B}$ by setting

$$d(A, B) := \mu(A \triangle B),$$

where $A \triangle B$ denotes the symmetric difference of A and B, that is,

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

The relation $d(A, B) = 0$ defines an equivalence relation $\sim_\mu$ on $\mathcal{B}$. We denote the equivalence class of A by $[A]_\mu$. The pseudometric d induces a metric on the set of equivalence classes by setting $d([A]_\mu, [B]_\mu) = d(A, B)$. Prove that the resulting metric space, that is, the set of $\sim_\mu$-equivalence classes with the aformentioned metric, is complete.

Problem 4

Does there exist a Lebesgue measurable subset $E \subset [0,1]$ such that $m(E \cap [0,a]) = \frac{a}{2}$ for all $a \in [0,1]$? Justify your answer.

Problem 5

Suppose that $f_n : [0,1] \to [0,\infty)$ is a sequence of measurable functions such that $\|f_n\|_2 \leq 1$ for all $n \in \mathbb{N}$. Further suppose that $f_n \to f$ a.e. Prove that:

(i) $f \in L^2[0,1]$.

(ii) $f_n \to f$ in $L^1[0,1]$.

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Problem 6

Let $f \in L^4[0,1]$ be such that $\|f\|_4 \leq 2\|f\|_2$. Show that $\|f\|_2 \leq 4\|f\|_1$.