Qualifying Examination

Math Exam ID Number: ____________________

The exam consists of six problem. Solve as many as you can by giving complete arguments and providing explicit justification. Show all your work. Each problem is worth 5 points.

Problem	**Score**
1	
2	
3	
4	
5	
6	
Total	

Problem 1

Let m be the Lebesgue measure on $(\mathbb{R}, \mathcal{L})$ and ν be the counting measure. Show that ν does not have a Lebesgue decomposition with respect to m.

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Problem 2

Let $E \subset \mathbb{R}$ be a not necessarily measurable set with $m^*(E) > 0$, where m^* is the Lebesgue outer measure given by

$$m^*(E) = \inf \left\{ \sum_{n \in \mathbb{N}} l(I_n) \,\middle|\, I_n \text{ is an open interval for } n \in \mathbb{N} \text{ and } E \subset \bigcup_{n \in \mathbb{N}} I_n \right\}$$

and $l(I)$ denotes the length of an interval $I \subset \mathbb{R}$. Show that

$$\forall \alpha \in (0,1)\, \exists I, \text{ open interval, s.t. } m^*(E \cap I) > \alpha\, m(I).$$

Problem 3

Let $(X, \mathcal{M}, \mu)$ be a measure space and prove that there exists a function $f \in L^1(X, \mu)$ satisfying $\mu\big([f = 0]\big) = 0$ if and only if $(X, \mathcal{M}, \mu)$ is σ-finite.

Problem 4

Let $(X, \mathcal{M}, \mu)$ be a probability space, that is, a measure space with $\mu(X) = 1$. Let f be a strictly positive $\mathcal{M}$-measurable function on X. Prove that

$$1 \leq \left(\int_X f \, d\mu \right) \left(\int_X \frac{1}{f} \, d\mu \right).$$

Problem 5

Suppose that $f \in L^1(\mathbb{R})$ and prove that

$$\lim_{n \to +\infty} f(n^2 x) = 0 \quad \text{for a.e. } x \in \mathbb{R}.$$

Problem 6

Let A and B be two measurable subsets of $\mathbb{R}$ with finite measures and define

$$f(x) = m\big((A - x) \cap B\big)$$

for the Lebesgue measure m. Prove that

$$\int_{\mathbb{R}} f(x)\, dx = m(B)m(A),$$

justifying all the steps in your calculation.