Real Analysis Qualifying Exam Spring 2018

June 19, 2018

Student’s math exam ID#: ____________________________

INSTRUCTIONS: Do all work on the sheets provided. There is a blank page following each problem. Please do not use the back of the sheets in your solutions.

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**Problem 1** (10 points) Let \( f \in L^2(\mathbb{R}) \), and let \((\alpha_n)_{n=1}^\infty, (\beta_n)_{n=1}^\infty \) be sequences of real numbers such that \( \alpha_n \neq 0 \) for all \( n = 1, 2, \ldots \), and \( \sum_{n=1}^\infty \frac{|\beta_n|}{|\alpha_n|^{1/2}} < \infty \). Show that \( \lim_{n \to \infty} \beta_n f(\alpha_n x) = 0 \) for almost all \( x \in \mathbb{R} \).
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Problem 2 (10 points) Let \( g_k, g \in L^1(\mathbb{R}) \) and assume that \( g_k \to g \) in \( L^1(\mathbb{R}) \) as \( k \to \infty \). Let \( (\alpha_k)_{k=1}^\infty \) be a bounded sequence in \( \mathbb{R} \), and let \( f_k(x) := g_k(x + \alpha_k) \), \( k = 1, 2, \ldots \). Prove that the sequence of functions \( (f_k)_{k=1}^\infty \) has a subsequence that converges in \( L^1(\mathbb{R}) \), almost everywhere, and in measure.
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Problem 3 (10 points) Let $f, g \in L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. Assume that for every $0 < t < \infty$,

$$m(\{x \in \mathbb{R}^n : |g(x)| > t\}) \leq \frac{1}{t} \int_{\{x \in \mathbb{R}^n : |g(x)| > t\}} |f(x)| dx,$$

where $m$ is the Lebesgue measure. Show that

$$\|g\|_{L^p(\mathbb{R}^n)} \leq p'\|f\|_{L^p(\mathbb{R}^n)}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$
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Problem 4 (10 points) Let $f \in L^1(\mathbb{R})$ be such that

$$\int_E f(y)dy = 0$$

for all Lebesgue measurable sets $E \subset \mathbb{R}$ with $m(E) = \pi$. Prove or disprove that $f(x) = 0$ for almost all $x \in \mathbb{R}$. 
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Problem 5 (10 points) Let $\mu$ be a positive finite Borel measure on $[0, 1]$ and let $\varphi : [0, 1] \to [0, 1]$ be continuous. Assume that $\mu(\varphi^{-1}(E)) = 0$ for every Borel set $E \subset [0, 1]$ with $\mu(E) = 0$. Show that there is a Borel measurable function $f : [0, 1] \to [0, \infty)$ such that

$$\int_0^1 g(\varphi(x))d\mu(x) = \int_0^1 g(x)f(x)d\mu(x)$$

for all continuous $g : [0, 1] \to \mathbb{R}$. 
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Problem 6 (10 points) Let $f \in L^p([0,1])$ for some $1 \leq p < \infty$. Compute

$$\lim_{n \to \infty} \left( n^{p-1} \sum_{k=0}^{n-1} \left( \int_{k/n}^{(k+1)/n} |f(y)| dy \right)^p \right)^{1/p}.$$ 

Explain fully.
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