Real Analysis Qualifying Exam Spring 2018

June 19, 2018

Student's math exam ID#: ______________________________

INSTRUCTIONS: Do all work on the sheets provided. There is a blank page following each problem. **Please do not use the back of the sheets in your solutions**.

Problem	Point Value	Points Received
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	**60**	

Student's math exam ID#: ______________________________

Problem 1 (10 points) Let $f \in L^2(\mathbb{R})$, and let $(\alpha_n)_{n=1}^{\infty}$, $(\beta_n)_{n=1}^{\infty}$ be sequences of real numbers such that $\alpha_n \neq 0$ for all $n = 1, 2, \ldots$, and $\sum_{n=1}^{\infty} \frac{|\beta_n|}{|\alpha_n|^{1/2}} < \infty$. Show that $\lim_{n\to\infty} \beta_n f(\alpha_n x) = 0$ for almost all $x \in \mathbb{R}$.

Student's math exam ID#: ______________________________

Student's math exam ID#: ______________________________

Problem 2 (10 points) Let $g_k, g \in L^1(\mathbb{R})$ and assume that $g_k \to g$ in $L^1(\mathbb{R})$ as $k \to \infty$. Let $(\alpha_k)_{k=1}^{\infty}$ be a bounded sequence in $\mathbb{R}$, and let $f_k(x) := g_k(x + \alpha_k)$, $k = 1, 2, \ldots$. Prove that the sequence of functions $(f_k)_{k=1}^{\infty}$ has a subsequence that converges in $L^1(\mathbb{R})$, almost everywhere, and in measure.

Student's math exam ID#: ____________________________

Student's math exam ID#: ________________________________

Problem 3 (10 points) Let $f, g \in L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. Assume that for every $0 < t < \infty$,

$$m(\{x \in \mathbb{R}^n : |g(x)| > t\}) \leq \frac{1}{t} \int_{\{x \in \mathbb{R}^n : |g(x)| > t\}} |f(x)| dx,$$

where m is the Lebesgue measure. Show that

$$\|g\|_{L^p(\mathbb{R}^n)} \leq p' \|f\|_{L^p(\mathbb{R}^n)}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Student's math exam ID#: ________________________________

Student's math exam ID#: ______________________________

Problem 4 (10 points) Let $f \in L^1(\mathbb{R})$ be such that

$$\int_E f(y)dy = 0$$

for all Lebesgue measurable sets $E \subset \mathbb{R}$ with $m(E) = \pi$. Prove or disprove that $f(x) = 0$ for almost all $x \in \mathbb{R}$.

Student's math exam ID#: ______________________________

Student's math exam ID#: ______________________________

Problem 5 (10 points) Let μ be a positive finite Borel measure on $[0,1]$ and let $\varphi : [0,1] \to [0,1]$ be continuous. Assume that $\mu(\varphi^{-1}(E)) = 0$ for every Borel set $E \subset [0,1]$ with $\mu(E) = 0$. Show that there is a Borel measurable function $f : [0,1] \to [0,\infty)$ such that

$$\int_0^1 g(\varphi(x))d\mu(x) = \int_0^1 g(x)f(x)d\mu(x)$$

for all continuous $g : [0,1] \to \mathbb{R}$.

Student's math exam ID#: ______________________________

Student's math exam ID#: ___________________________

Problem 6 (10 points) Let $f \in L^p([0,1])$ for some $1 \le p < \infty$. Compute

$$\lim_{n\to\infty} \left(n^{p-1} \sum_{k=0}^{n-1} \left(\int_{k/n}^{(k+1)/n} |f(y)|dy \right)^p \right)^{1/p}.$$

Explain fully.

Student's math exam ID#: ______________________________