Algebra Qualifying Exam, June 2017

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Each of the 10 questions is worth 10 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators. As usual, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers respectively, $\mathbb{F}_q$ denotes a finite field with q elements, and S_n denotes the symmetric group on n elements.

1. Suppose G is a group of order 80. Prove that G is not simple.

2. Prove that the additive group $\mathbb{R}/\mathbb{Z}$ is isomorphic to the multiplicative group $\{z \in \mathbb{C} : |z| = 1\}$.

3. Let R be an integral domain. A nonzero nonunit element $p \in R$ is *prime* if $p \mid ab$ implies that $p \mid a$ or $p \mid b$. A nonzero nonunit element $p \in R$ is *irreducible* if $p = ab$ implies that a or b is a unit.
   (a) Show that every prime element is irreducible.
   (b) Show that if R is a Unique Factorization Domain (UFD), then every irreducible element is prime.

4. Let G and H be finite abelian groups, and suppose that the order of G is relatively prime to the order of H. Show that $G \otimes_{\mathbb{Z}} H = 0$.

5. Let D_8 be the dihedral group of order 8.
   (a) Compute the center of D_8.
   (b) Compute the commutator subgroup of D_8.
   (c) Compute the conjugacy classes of D_8.

6. Let R be a commutative ring with 1, and let M be an R-module. Show that if $M \oplus M$ is a finitely generated R-module, then M is a finitely generated R-module.

7. (a) Find a polynomial $f(x) \in \mathbb{Q}[x]$ whose splitting field L_f has Galois group $\mathrm{Gal}(L_f/\mathbb{Q})$ isomorphic to $\mathbb{Z}/2\mathbb{Z}$.
   (b) Find a polynomial $g(x) \in \mathbb{Q}[x]$ whose splitting field L_g has Galois group $\mathrm{Gal}(L_g/\mathbb{Q})$ isomorphic to S_3.
   (c) Find a polynomial $h(x) \in \mathbb{Q}[x]$ whose splitting field L_h has Galois group $\mathrm{Gal}(L_h/\mathbb{Q})$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times S_3$.

   Justify your answers.

8. Suppose F is a field and $f(x) \in F[x]$ is a nonconstant polynomial. Show that $F[x]/(f(x))$ is a direct product of fields if and only if $f(x)$ is a separable polynomial.

9. Suppose p is a prime.
   (a) Show that all matrices $A \in \mathrm{GL}_2(\mathbb{F}_p)$ of order exactly p have the same characteristic polynomial, and find that polynomial.
   (b) Show that all matrices $A \in \mathrm{GL}_2(\mathbb{F}_p)$ of order exactly p have the same minimal polynomial, and find that polynomial.

10. Let $K = \mathbb{F}_3(\sqrt{2})$ and let $f(x) = x^4 + 1 \in \mathbb{F}_3[x]$.
    (a) Show that K is the splitting field of f.
    (b) Find a generator α of the multiplicative group $K^\times$.
    (c) Express the roots of f in terms of α.