Instructions *Complete all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions.*

1. Let A be a subset of $\mathbb{R}$ of positive Lebesgue measure. Prove that there exist $k, n \in \mathbb{N}$ and $x, y \in A$ with $|x - y| = \frac{k}{2^n}$.

2. Either prove or give a counterexample:

   If a sequence of functions f_n on a measure space (X, μ) satisfies $\int_x |f_n| d\mu \le \frac{1}{n^2}$, then $f_n \to 0$ μ-a.e.

3. Let $f \in L^4([a, b])$, and let $F(x) = \int_a^x f(x)dx$. Show that $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h^{3/4}} = 0$ for all $x \in (a, b)$.

4. Assume $f, g \in L^2(\mathbb{R})$. Define

   $$A(x) = \int_{\mathbb{R}} f(x - y)g(y)\, dy.$$

   Show that $A(x) \in C(\mathbb{R})$ and

   $$\lim_{|x|\to+\infty} A(x) = 0.$$

5. Is it possible for a continuous function $f : [0, 1] \to \mathbb{R}$ to have

   (a) Infinitely many strict local minima?

   (b) Uncountably many strict local minima?

   Prove your answers.

6. Let A be the collection of functions $f \in L^1(X, \mu)$ such that $\|f\|_1 = 1$ and $\int_X f d\mu = 0$. Prove that for every $g \in L^\infty(X, \mu)$,

   $$\sup_{f\in A} \int_X fg d\mu = \frac{1}{2}(\operatorname{ess\,sup} g - \operatorname{ess\,inf} g)$$

   .

   If you haven't seen ess inf before, it is defined in a natural way, or else as

   $$\operatorname{ess\,inf} f = -\operatorname{ess\,sup} -f.$$