Real Analysis Qualifying Exam 2015 Spring: *complete all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions.*

Problem 1: Show that if $f \in L^4(\mathbb{R})$ then

$$\lim_{c\to 1} \int_{\mathbb{R}} |f(cx) - f(x)|^4 dx = 0.$$

Problem 2: Let $f_n : (0,\infty) \to \mathbb{R}$, $n = 1, 2, \ldots$, be a sequence of Lebesgue measurable functions such that $f_n \to f$ a.e. as $n \to \infty$. Assume that there exists $g : (0,\infty) \to \mathbb{R}$ such that $|f_n| \le g$, $n = 1, 2, \ldots$, and $g \in L^1((0,a))$ for all $0 < a < \infty$. Assume furthermore that

$$\int_1^\infty |f_n(\sqrt{x})| dx \le C, \quad n = 1, 2, \ldots,$$

for some constant $C > 0$. Show that $f_n \in L^1((0,\infty))$, $f \in L^1((0,\infty))$ and $f_n \to f$ in $L^1((0,\infty))$ as $n \to \infty$.

Problem 3: Assume that $f \in C^1((0,1))$ and

$$\int_{(0,1)} x|f'|^p \, dx < +\infty \quad \text{for some } p > 2.$$

Show that $\lim_{x\to 0^+} f(x)$ exists.

Problem 4: Suppose that $E \subset [0,1]^2$ is measurable. Denote

$$E_x = \{y \in [0,1] : \ (x,y) \in E\} \quad \text{and} \quad E_y = \{x \in [0,1] : \ (x,y) \in E\}.$$

Show that if $m(E_x) = 0$ for a.e $x \in [0, \frac{1}{2}]$, then

$$m(\{y \in [0,1] : \ m(E_y) = 1\}) \le \frac{1}{2}.$$

Problem 5: Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$, and let $\alpha > 1 - \frac{1}{p}$. Show that the series

$$\sum_{n=1}^{\infty} \int_n^{n+n^{-\alpha}} |f(x+y)| dy$$

converges for a.e. $x \in \mathbb{R}$.

Problem 6: Suppose $E \subset \mathbb{R}$ is measurable and $E = E + \frac{1}{n}$ for every natural number $n \ge 1$. Show that either $m(E) = 0$ or $m(E^c) = 0$.