ALGEBRA QUALIFYING EXAM

FALL 2016

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Each of the 10 questions is worth 10 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

Notation: For this exam, all rings have 1 and all ring homomorphisms send 1 to 1.

1. Is every group of order 39 cyclic? Either prove this or construct a non-cyclic group of order 39.
2. Recall that for H a subgroup of G, the normalizer $N(H)$ is defined to be $\{g \in G : gHg^{-1} = H\}$. Let p be a prime and S_p denote the symmetric group on p letters. Let H be a subgroup of $G = S_p$ of order p. What is the order of $N(H)$? Explain your answer.
3. Prove or disprove: The quotient ring $\mathbb{F}_2[x]/(x^4 + x^3 + x^2 + x + 1)$ is a field.
4. Does there exist a ring with 1 whose additive group is isomorphic to $\mathbb{Q}/\mathbb{Z}$? Prove your answer.
5. Let R denote a commutative ring with 1 such that for every $r \in R$, we have $r^n = r$ for some integer $n > 1$. Prove that every prime ideal in R is maximal.
6. Let $\zeta = e^{2\pi i/7} \in \mathbb{C}$ denote a primitive 7-th root of unity.
   (a) True/False: Every element in $\mathbb{Q}(\zeta)$ can be expressed <u>uniquely</u> in the form
   $$a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 + a_5\zeta^5 + a_6\zeta^6,$$
   where $a_0, \ldots, a_6 \in \mathbb{Q}$. Briefly explain.
   (b) Find the order of the element $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ induced by $\sigma : \zeta \mapsto \zeta^2$. Briefly explain your answer.
   (c) Find the degree of the field extension $\mathbb{Q}(\zeta + \zeta^2 + \zeta^4)/\mathbb{Q}$. Explain your answer.
7. Give an example of a 10×10 matrix over $\mathbb{R}$ with minimal polynomial $(x^4 - 2)(x + 2)^2$ which is not similar to any matrix with rational entries. Briefly explain your answer.
8. Let D_{10} denote the dihedral group of order 10.
   (a) Give an example of a non-trivial degree one representation $D_{10} \to \mathrm{GL}_1(\mathbb{R})$.
   (b) Give an example of an irreducible degree two representation $D_{10} \to \mathrm{GL}_2(\mathbb{R})$. Prove that your representation is irreducible.
9. True/False. For each of the following answer True or False and give a <u>brief</u> explanation.
   (a) Every finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$ is abelian.
   (b) A finite extension of $\mathbb{Q}$ cannot have infinitely many distinct subfields.
10. For each of the following, either give an example or state that none exists. In either case, give a <u>brief</u> explanation.
    (a) A non-zero zero divisor in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.
    (b) An injective group homomorphism $(\mathbb{Z}/8\mathbb{Z})^\times \to \mathbb{Z}/24\mathbb{Z}$.