Real Analysis Qualifying Exam 2016 Fall

Your Name: ____________________________________

INSTRUCTIONS: *Complete all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions.*

Problem	Point Value	Points Received
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

Your Name: ____

Problem 1: Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and let $\mathcal{A}_0$ be a sub-σ-algebra of $\mathcal{A}$. Given an a nonnegative $\mathcal{A}$-measurable function f on X, show that there is a nonnegative $\mathcal{A}_0$-measurable function f_0 on X, such that

$$\int_X fg \, \mathrm{d}\mu = \int_X f_0 g \, \mathrm{d}\mu$$

for every nonnnegative $\mathcal{A}_0$-measurable function g. In what sense is f_0 unique?

Your Name: ___________________________________

Problem 2: Let $(X, \mathcal{A}, \mu)$ be a measure space. Suppose there exists an extended real valued $\mathcal{A}$- measurable function f on X such that $f > 0$ μ-a.e. and f is μ-integrable. Prove that $(X, \mathcal{A}, \mu)$ is a σ-finite measure space.

Your Name: ____________________________________

Problem 3: Let $0 \leq f \in L^1(\mathbb{R})$. Set $f_1 = f$ and $f_{n+1} = f * f_n$ for $n = 1, 2, \ldots$.

(i) Show that the series $\sum_{n=1}^{\infty} f_n$ converges in $L^1(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} f dx < 1.$$

(ii) Suppose that $f(x) = 0$ for $x < 0$ and that f is bounded for $x < A$, for some $A > 0$. Show that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly for $x < A$.

Your Name: ________________________________

Problem 4: Construct a nonnegative measureable function f on $[0, 1]$ such that

(i) $f \in L^m([0, 1])$ for any $m > 0$;

(ii) $\text{esssup}_I f = +\infty$ for any interval $I \subset [0, 1]$.

Your Name: ___________________________________

Problem 5: Assume that f is a nonnegative measurable function on $[0,1]$ and

$$\int_{[0,1]} f\,dx = \int_{[0,1]} f^2\,dx = \int_{[0,1]} f^3\,dx.$$

Show that $f(1-f)=0$ a.e.

Your Name: ________________________________

Problem 6: Let (f_n) be a sequence of measurable functions on $\mathbb{R}^d$ such that $|f_n(x)| \leq 1$ for all x and all n and assume that

$$f_n \to f \quad \text{a.e.,} \quad \text{as} \quad n \to \infty.$$

Show that $g * f_n \to g * f$ uniformly on each compact set in $\mathbb{R}^d$, if $g \in L^1(\mathbb{R}^d)$.