Math 2A: Sample Final 3

• Turn off your cell phone and do not check it during the exam.
• No calculators or other forms of assistance allowed.
• This exam consists of 14 questions for 100 total points. Points per question are in brackets.
• Read the directions for each problem carefully and answer all parts of each problem.
• Unless instructed otherwise, show all work for full credit.
• Define any notation used and label any sketches/graphs.
1. Compute the following derivatives:

(a) \( f'(x) \) where \( f(x) = \sin(x^2 + 2) \)

(b) \( \frac{d^2s}{dt^2} \) where \( s = 25t + t^{3/2} \)

(c) \( \frac{d}{dz} \tan^{-1}(\sqrt{z}) \)

(d) \( f'(x) \) where \( f(x) = x^{\sin x} \)
2. Find the equation of the tangent line to the curve \( x^3 + xy + y^2 = 1 \) at the point \((-1, 2)\).

3. Show that the equation \( 2^{-x} = \frac{x - 1}{x + 1} \) has at least one solution in the interval \([1, 3]\). Explain your answer and state what theorem you are using.
4. Compute the limits.

(a) \[ \lim_{x \to 3^+} \frac{2 - x}{\sqrt{x - 3}} \]

(b) \[ \lim_{x \to \infty} \frac{x^{1/3}}{\ln x} \]

5. An equilateral triangle \( \triangle ABC \) (see picture) initially has sides of length 10 in. Suppose that the side lengths shrink at a constant rate of 2 in/sec.

(You may use the fact that the area of an equilateral triangle with side length \( \ell \) is \( \frac{\sqrt{3}}{4} \ell^2 \))

How rapidly is the area of the triangle changing when its area is \( 4\sqrt{3} \) in\(^2\)?
6. Use a linear approximation (or differentials) to estimate $\sqrt[3]{9}$.
   (Hint: Let $f(x) = \sqrt[3]{x}$ and recall that $\sqrt[3]{8} = 2$)

7. Find the anti-derivative $F(x)$ of the function $f(x) = 3x^2 + e^x$ which satisfies $F(1) = e$. 

8. You are given the following graphs for functions $f$ and $g$.

(a) State the following limits. Write DNE if the limit does not exist. (No working required)

i. $\lim_{x \to 3} f(x)$  

(ii. $\lim_{x \to 3^+} \frac{f(x)}{g(x)}$

(b) Define the function $h(x) = 2f(x) - 3g(x)$. Is $h$ continuous or discontinuous at $x = 3$? Explain.
9. Consider the function \( f(x) = 2xe^{-x} \) with domain \([0, \infty)\).

(a) Compute the first two derivatives \( f'(x) \) and \( f''(x) \).

(b) Complete the following table. State ‘none’ if applicable. No working necessary.

<table>
<thead>
<tr>
<th>Absolute/local minima</th>
<th>Absolute/local maxima</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interval(s) of increase</td>
<td>Interval(s) of decrease</td>
</tr>
<tr>
<td>Inflection point(s)</td>
<td></td>
</tr>
<tr>
<td>Interval(s) of upwards concavity</td>
<td>Interval(s) of downwards concavity</td>
</tr>
<tr>
<td>Horizontal Asymptote(s)</td>
<td>Vertical Asymptote(s)</td>
</tr>
</tbody>
</table>

(c) Sketch the curve \( y = f(x) \) on the axes below.
10. Use the limit definition to compute the derivative of \( f(x) = \frac{1}{x} \)

11. The profit made by EZ-motors when it sells \( n \) cars is given by a function \( P(n) \). Suppose, when \( n = 100 \), that \( dP = $1,000 \). Which of the following sentences must be true? Circle all that apply.

   (a) If EZ-motors sells 102 cars, then it will make exactly $2,000 profit.
   (b) If EZ-motors sells 97 cars, then it will make approximately $3,000 less profit than if it sold 100 cars.
   (c) If \( P(100) = $50,000 \), then \( P(104) \approx $54,000 \).
   (d) The rate of change of \( P(n) \) is \( \frac{1000}{100} = 10 \$ / \text{car} \).
12. Answer true or false to each of the following and give a short explanation. Unjustified answers will receive no credit.

(a) \[ \lim_{x \to -\infty} \frac{\sqrt{x^6 + 5x + 9}}{3x^3 + 2} = \frac{1}{3} \] 

(b) Let \( f(x) = x^2 \). Then there exists some \( c \in (2, 4) \) for which \( f'(c) = 6 \).

13. The rate of change of the temperature (\( T \) °F) of a cup of coffee is proportional to the temperature difference between the coffee and the surrounding air (70°F). Suppose, at time \( t = 0 \) minutes, that the coffee has temperature 180°F, and is decreasing at a rate of 5°F/min. Which of the following is a correct model for the temperature of the coffee? (No working necessary)

(a) \( \frac{dT}{dt} = -22(T - 70) \)
(b) \( \frac{dT}{dt} = 22(T - 70) \)
(c) \( \frac{dT}{dt} = -\frac{1}{22}(T - 70) \)
(d) \( \frac{dT}{dt} = \frac{1}{22}(T - 70) \)
14. A cylinder has base radius \( r \) and height \( h \). Its volume and surface area are given by the formulas

\[
V = \pi r^2 h \quad \text{and} \quad A = 2\pi r(r + h)
\]

Find the dimensions of the cylinder with maximum volume if its surface area is \( A = 6\pi \). (8)