ALGEBRA QUALIFYING EXAM

SPRING 2016

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Each of the 10 questions is worth 10 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

1. (a) Prove that every subgroup of a cyclic group is cyclic.
   (b) Is the automorphism group of a cyclic group necessarily cyclic? Explain.
2. Let $G = \mathbb{Z}/25\mathbb{Z}$, the cyclic group of order 25.
   (a) Can G be given the structure of a $\mathbb{Z}[i]$-module? Explain your answer.
   (b) Can G be given the structure of a $\mathbb{Z}/5\mathbb{Z}$-module? Explain your answer.
3. Prove there is no simple group of order 520.
4. Let G be a finite group acting transitively on a set X with $|X| > 1$.
   (a) State the Orbit-Stabilizer Theorem.
   (b) Show that there is some element of G which fixes no element of X.
5. Let K be a field and let $\overline{K}$ be an algebraic closure of K. Assume $\alpha, \beta \in \overline{K}$ have degree 2 and 3 over K, respectively.
   (a) Can $\alpha\beta$ have degree 5 over K? Either give an example or prove this is impossible.
   (b) Can $\alpha\beta$ have degree 6 over K? Either give an example or prove this is impossible.
6. Let $L/\mathbb{Q}$ denote a Galois extension with Galois group isomorphic to A_4.
   (a) Does there exist a quadratic extension $K/\mathbb{Q}$ contained in L? Prove your answer.
   (b) Does there exist a degree 4 polynomial in $\mathbb{Q}[x]$ with splitting field equal to L? Prove your answer.
7. Let $A : V \to V$ be a linear transformation of a vector space V over the field $\mathbb{Q}$ which satisfies the relation $(A^3 + 3I)(A^3 - 2I) = 0$. Show that the dimension $\dim_{\mathbb{Q}}(V)$ is divisible by 3.
8. True/False. For each of the following answer True or False and give a brief explanation.
   (a) If K_1, K_2 are fields and $\varphi : K_1 \to K_2$ is a ring homomorphism such that $\varphi(1) = 1$, then φ is injective.
   (b) The unit group of $\mathbb{C}$ is isomorphic to the additive group of $\mathbb{C}$.
   (c) Let n be a positive integer. Then $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.
9. For each of the following, either give an example or state that none exists. In either case, give a brief explanation.
   (a) An element $\alpha \in \mathbb{Q}(\sqrt{2}, i)$ such that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, i)$.
   (b) A tower of field extensions $L \supseteq K' \supseteq K$ such that L/K' and K'/K are Galois extensions but L/K is not Galois.
10. Let $L_1, \ldots, L_r$ be all pairwise non-isomorphic complex irreducible representations of a group G of order 12. What are the possible values for their dimensions $n_i = \dim_{\mathbb{C}} L_i$? For each of the possible answers of the form $(n_1, \ldots, n_r)$ give an example of G which has such irreducible representations.