Algebra Qualifying Exam, June 2013

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Each of the 10 questions is worth 10 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators. As usual, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers respectively.

1. Let G be a group of order $2pq$, where p and q are odd primes, not necessarily distinct. Show that G is solvable.

2. Give an example of each of the following, and justify your answers:

   (a) a prime ideal that is not maximal, in a commutative ring with identity;

   (b) two commutative rings R_1, R_2 whose underlying additive groups are isomorphic, but such that R_1 and R_2 are not isomorphic as rings;

   (c) a Unique Factorization Domain that is not a Principal Ideal Domain.

3. (a) For which $n \in \mathbb{N}$ does the permutation group S_n contain a subgroup isomorphic to $\mathbb{Z}/7\mathbb{Z}$?

   (b) For which $n \in \mathbb{N}$ does S_n contain a subgroup isomorphic to $\mathbb{Z}/14\mathbb{Z}$?

4. Let R be a product of two commutative rings S and T with unity.

   (a) Prove that all ideals of R are of the form $I \times J$ for two ideals I of S and J of T, respectively.

   (b) What do prime ideals and maximal ideals of R look like?

5. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible cubic polynomial whose Galois group is denoted by G_f.

   (a) Prove that if $f(x)$ has exactly one real root, then $G_f \cong S_3$.

   (b) Find an irreducible cubic $f(x) \in \mathbb{Q}[x]$ whose roots generate the cubic subextension of $\mathbb{Q}(\zeta_7)/\mathbb{Q}$, where ζ_7 denotes a primitive 7-th root of unity in $\mathbb{C}$.

6. Let E be the splitting field of $x^{35} - 1$ over $\mathbb{F}_2$.

   (a) How many elements does E have?

   (b) How many subfields does E have?

7. Let V be a finite dimensional vector space of dimension n over $\mathbb{Q}$. Let $A \in \mathrm{End}_{\mathbb{Q}}(V)$ be a linear map from V to itself. Assume that $A^7 = I_n$. Assume further that A has no non-zero fixed points in V. Show that the dimension n is divisible by 6.

8. Suppose R is a commutative ring with identity and M_1, M_2 are distinct maximal ideals of R. Show that $R/M_1 \otimes_R R/M_2 = 0$.

9. Suppose that K is a Galois extension of F of degree pq, where p and q are distinct primes and $q > p$. Show that K has a subfield L Galois over F such that $[L : F] = p$.

10. Let G be a finite cyclic p-group and let $\rho : G \longrightarrow \mathrm{Aut}_F(V)$ be a representation on a finite dimensional vector space V over a field F of charactersitic p. Assume that ρ is irreducible. Prove that ρ is trivial, i.e., G acts trivially on V.