ALGEBRA QUALIFYING EXAM
Fall 2015

Instructions: LABEL YOUR ANSWERS CLEARLY. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

Notation: Let $\mathbb{F}_q$ denote the finite field with $q$ elements. Let $\mathbb{Z}$ denote the integers. Let $\mathbb{Q}$ denote the rational numbers. Let $\mathbb{R}$ denote the real numbers. Let $\mathbb{C}$ denote the complex numbers.

1. (a) Define **prime ideal**.
   (b) Define **maximal ideal**.
   (c) Give an example of a ring $R$ and ideals $P_1$, $P_2$, and $P_3$ of $R$ such that for the properties “prime ideal” and “maximal ideal” of $R$,
   i. $P_1$ satisfies both properties,
   ii. $P_2$ satisfies neither property,
   iii. $P_3$ satisfies one property but not the other.

   Justify your answers.

2. Show that if a group $G$ has only finitely many subgroups then $G$ is a finite group.

3. Let $A$ be an $n \times n$ matrix with entries in $\mathbb{R}$ such that $A^2 = -I$.
   (a) Prove that $n$ is even.
   (b) Prove that $A$ is diagonalizable over $\mathbb{C}$ and describe the corresponding diagonal matrices.

4. Let $G$ be a group of order 70. Prove that $G$ has a normal subgroup of order 35.

5. Construct a Galois extension $F$ of $\mathbb{Q}$ satisfying $\text{Gal}(F/\mathbb{Q}) \simeq D_8$, the dihedral group of order 8. Fully justify.

6. Let $F$ be a field. Prove that every ideal of $F[x]$ is principal.

7. Give an example of a module $M$ over a ring $R$ such that $M$ is **not** finitely generated as an $R$-module.
   Prove that it is not finitely generated as an $R$-module.

8. Suppose $H$ is a normal subgroup of a finite group $G$.
   (a) Prove or disprove: If $H$ has order 2, then $H$ is a subgroup of the center of $G$.
   (b) Prove or disprove: If $H$ has order 3, then $H$ is a subgroup of the center of $G$.

9. (a) What does it mean for a representation to be **irreducible**?
   (b) Suppose $p$ is a prime. Let $G = \mathbb{Z}/p\mathbb{Z}$ and let $\rho : G \to \text{GL}_2(\mathbb{F}_p)$ be a representation. Show that $\rho$ is reducible.

10. (a) Compute the order of $\text{GL}_4(\mathbb{F}_3)$. (Justify your reasoning.)
    (b) Compute the order of $\text{SL}_4(\mathbb{F}_3)$. (Justify your reasoning.)
    (c) Show that $\mathbb{Z}[\sqrt{10}]$ is not a UFD.