# ALGEBRA QUALIFYING EXAM

September 19, 2012

**Instructions:** JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Each of the 10 questions is worth 10 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

*Notation:* Let $\mathbb{F}_q$ denote the finite field with $q$ elements, $\mathbb{Z}$ the integers, $\mathbb{Q}$ the rational numbers, and $\mathbb{R}$ the real numbers. Let $S_n$ denote the symmetric group on $n$ letters. If $R$ is a field, then $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices with entries in $R$, and $\mathrm{SL}_n(R)$ is the subgroup of matrices in $\mathrm{GL}_n(R)$ with determinant 1.

1. Suppose $R$ is a commutative ring. Recall that an element $r \in R$ is *nilpotent* if $r^n = 0$ for some positive integer $n$. Show that if $r, s \in R$ are nilpotent, then $r + s$ is nilpotent. Is this still true if we remove the requirement that $R$ is commutative? Justify your answer.

2. (a) How many Sylow 5-subgroups are there in $S_5$?
   (b) Find a Sylow 2-subgroup of $S_5$. Describe it explicitly as an abstract group and as a collection of permutations in $S_5$.

3. Consider the following two properties of a module $M$ over the ring $\mathbb{Z}[x]$.

   (*) For every sequence $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of submodules of $M$, there exists an $i$ with $M_i = M_{i+1} = M_{i+2} = \cdots$.
   (**) Every submodule $M' \subseteq M$ is finitely generated.

   (a) Does (*) imply (**)? Prove or disprove.
   (b) Does (**) imply (*)? Prove or disprove.

4. Show that the squares in a group $G$ are contained in every subgroup of index 2. Must the cubes in $G$ be contained in every subgroup of index 3?

5. Suppose $F$ is a Galois extension of $\mathbb{Q}$ and $\mathrm{Gal}(F/\mathbb{Q}) \cong S_4$. Show that there is an irreducible polynomial $g(x) \in \mathbb{Q}[x]$ of degree 4 such that the splitting field of $g(x)$ is $F$.

6. How many conjugacy classes are there in the group $\mathrm{GL}_3(\mathbb{F}_2)$ of invertible $3 \times 3$ matrices over $\mathbb{F}_2$? (Hint: use rational canonical forms and/or the fundamental theorem of finitely generated modules over $\mathbb{F}_2[x]$.)

7. Let $G$ denote a finite group, let $K$ denote a field, and let $\varphi : G \to \mathrm{GL}_n(K)$ denote a representation.

   (a) Prove or disprove: $\varphi(G') \subseteq \mathrm{SL}_n(K)$, where $G'$ is the commutator subgroup of $G$.
   (b) Prove or disprove: $\varphi(Z(G)) \subseteq \mathrm{SL}_n(K)$, where $Z(G)$ is the center of $G$.

8. Determine the Galois group of the splitting field of the polynomial $x^3 + 2$ over $\mathbb{F}_3$, over $\mathbb{F}_7$, and over $\mathbb{F}_{11}$.

9. Short answer. For each of the following give the answer and a *brief* explanation.

   (a) True or False: If a group has an element of order $m$ and an element of order $n$, then it has an element of order $\mathrm{lcm}(m, n)$.
   (b) True or False: If $K/F$ is a Galois extension with cyclic Galois group, and $E$ is a field satisfying $F \subseteq E \subseteq K$, then $E/F$ is also Galois with cyclic Galois group.
   (c) True or False: There are at most $(n!)^n$ groups of order $n$, up to isomorphism.
   (d) What is the largest order of an element in $D_{64}$ (the group of symmetries of a 32-gon)?
   (e) If $V, W$ are finite dimensional vector spaces over a field $F$, what is the dimension of the tensor product $V \otimes_F W$?

10. For each of the following, either give an example or explain *briefly* why no such example exists:

    (a) a quadratic extension of fields that is not separable.
    (b) a nonabelian group in which all the proper subgroups are cyclic.
    (c) an infinite field where every nonzero element has finite multiplicative order.
    (d) a nonabelian group with trivial automorphism group.
    (e) an element of order 4 in $\mathbb{R}/\mathbb{Z}$.