Algebra Comprehensive Exam

June, 2014

NAME	
SIGNATURE	

- This is a closed-book test. You have 2 hours and 30 minutes to complete the exam.
- The test contains 10 problems. Each problem is worth 10 points.
- **Show all details and quote any theorem you use. We prefer complete solutions of a few problems to many partial solutions.**
- *Please, write clearly and legibly.* Clearly indicate scratch work so it won't be graded.

Problem	1	2	3	4	5	6	7	8	9	10
Score										

Problem 1

Suppose G is a finite subgroup, H is a subgroup, and $a \in G$ an element such that $a^k \in H$ for some integer k satistying $GCD(k, |G|) = 1$. Prove that a is in H.

Problem 2

Describe all permutations in S_n which commute with the n-cycle $(12\ldots n)$.

Problem 3

Classify all groups of order $2014 = 2 \cdot 19 \cdot 53$.

(HINT: show that there is a normal subgroup isomorphic to $\mathbb{Z}/19\mathbb{Z} \times \mathbb{Z}/53\mathbb{Z}$ and then observe that conjugation by an element of order two induces an order two automorphism of this subgroup.)

Problem 4

In this problem you will work with endomorphisms of $\mathbb{Z} \times \mathbb{Z}$. Consider the ring $\mathbb{Z} \times \mathbb{Z}$ with component-wise addition and multiplication. Write elements of $\mathbb{Z} \times \mathbb{Z}$ as column vectors:

$$\mathbb{Z} \times \mathbb{Z} = \left\{ \begin{pmatrix} m \\ n \end{pmatrix} : m, n \in \mathbb{Z} \right\}.$$

Now let $End(\mathbb{Z} \times \mathbb{Z})$ be the set of all ring homomorphisms from $\mathbb{Z} \times \mathbb{Z}$ to itself under point-wise addition and composition.

(a) Show that every endomorphism F of $\mathbb{Z} \times \mathbb{Z}$ is uniquely determined by the value of $F \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $F \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) Prove that the map

$$End(\mathbb{Z} \times \mathbb{Z}) \to M_2(\mathbb{Z}),\ F \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\begin{pmatrix} a \\ c \end{pmatrix} = F \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix} = F \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a ring isomorphism.

(c) Show that the set

$$R = \left\{ \begin{pmatrix} m & n \\ 0 & k \end{pmatrix} : m, n, k \in \mathbb{Z} \right\}$$

is a subring of $M_2(\mathbb{Z})$. Describe the corresponding subring of $End(\mathbb{Z} \times \mathbb{Z})$.

Problem 5

Let A and B be ideals of a commutative ring R.
(a) Show that $A \cap B$ and $A + B$ are ideals of R.
(b) If $R = \mathbb{Z}$, $A = (m)$ and $B = (n)$, describe the ideals $A \cap B$ and $A + B$.

Problem 6

Suppose that A is a nilpotent matrix. Show that $\det(A + I) = 1$.

Problem 7

Classify (up to conjugation by an orthogonal matrix) all matrices of size 4×4 with real coefficients which are orthogonal and skew symmetric ($A^t = -A$) at the same time.

Problem 8

Let A be a self-adjoint operator on a finite dimensional complex vector space V with a scalar product. Let v_1, v_2 be two eigenvectors of A with distinct eigenvalues λ_1, λ_2. Show that they are orthogonal.

Problem 9

Suppose that E/F is a finite extension of the form $E = F(r)$ with $[E : F]$ odd. Prove that $E = F(r^2)$.

Problem 10

Find all primes $p > 2$ for which the polynomial $x^2 + x + 1$ is irreducible in $\mathbb{F}_p[x]$.