Comprehensive Examination in Algebra

Spring 2016

Name: ____

Signature: ____

- The exam duration is 2 hours and 30 minutes.
- This is a closed-book test.
- There are 10 problems, each problem is worth 10 points.
- **Show all details and quote any theorem you use.** Only answers that explain every step will receive full credit.
- We prefer complete solutions to a few problems over many partial solutions.
- Please, write clearly and legibly. Clearly indicate scratch work so it will not be graded.

1	2	3	4	5	6	7	8	9	10	Total

1. Prove that the center of a p-group is never trivial.

2. Show that there is no simple group of order 56.

3. Describe all the maximal ideals in the following rings.
   (a) $\mathbb{C}[x]$
   (b) $\mathbb{Z}/n\mathbb{Z}$, where $n \in \mathbb{Z}_{>0}$.

4. Let R be a commutative ring. Show that an ideal I of R is maximal if and only if R/I is a field.

5. For which values of $a \in \mathbb{F}_5$ is the ring

$$\mathbb{F}_5[x]/\langle x^3 + ax + 2\rangle$$

a field?

6. Let W_1 and W_2 be linear subspaces of a finite dimensional vector space V. Show that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

7. Let $\mathbb{F}_q$ be a finite field of order q.
   (a) What is the order of $\mathrm{GL}_n(\mathbb{F}_q)$?
   (b) What is the order of $\mathrm{SL}_n(\mathbb{F}_q)$?

8. Let F be a field and K be an extension of F. Let $a, b \in K$. Prove that if a and b are algebraic over F, then $a + b$ is also algebraic over F.

9. Let $A \in \mathrm{Mat}_n(\mathbb{C})$. Prove that if A is nilpotent (i.e. there is a $k \in \mathbb{Z}_{>0}$ s.t. $A^k = 0$), then $\operatorname{Tr} A = 0$.

10. Prove that, for a matrix $A \in \mathrm{Mat}_n(\mathbb{R})$, the minimal and characteristic polynomial of A coincide if and only if there is a basis of $\mathbb{R}^n$ of the form $\{v, Av, A^2v, \ldots, A^{n-1}v\}$.