1. Prove that every finite group of order $> 2$ has a nontrivial automorphism.

2. In this problem there is no need to justify your answers.
   (a) Define UFD.
   (b) Define PID.
   (c) For the properties “UFD” and “PID”, give an example of a commutative integral domain that
      i. satisfies both properties,
      ii. satisfies one property but not the other,
      iii. satisfies neither property.

3. (a) Prove that $\mathbb{Q}((\sqrt{7}))$ is not Galois over $\mathbb{Q}(T)$, where $T$ is an indeterminate.
   (b) Find the Galois closure of $\mathbb{Q}((\sqrt{7}))$ over $\mathbb{Q}(T)$ and determine the Galois group both as an abstract
      group and as a set of explicit automorphisms. (Fully justify.)

4. Let $R$ be a commutative ring with multiplicative identity. An element $r \in R$ is called nilpotent if there
   exists a positive integer $n$ such that $r^n = 0$.
   (a) Prove that every nilpotent element lies in every prime ideal.
   (b) Assume that every element of $R$ is either nilpotent or a unit. Prove that $R$ has a unique prime ideal.

5. For every positive integer $n$, denote by $C_n$ a cyclic group of order $n$ and by $D_n$ a dihedral group of order
   $2n$, so that
   $$D_n = \{1, a, a^2, \ldots, a^{n-1}, b, ba, ba^2, \ldots, ba^{n-1}\}$$
   where $a$ has order $n$, $b$ has order 2 and $ab = ba^{-1}$.
   (a) In the notation explained above, prove that every subgroup of $\langle a \rangle$ is normal in $D_n$.
   (b) If $n = 2m$ with $m$ odd, prove that $D_n = D_{2m} \cong C_2 \times D_m$.
   (c) Is $D_{12} \cong C_3 \times D_4$? Justify your answer.

6. Suppose that $p$ and $q$ are prime numbers with $p < q$. Prove that no group of order $p^2q$ is simple.

7. Determine the maximal ideals of the following rings (fully justify):
   (a) $\mathbb{Q}[x]/(x^2 - 5x + 6),$
   (b) $\mathbb{Q}[x]/(x^2 + 4x + 6)$.

8. Find two matrices having the same characteristic polynomials and minimal polynomials but different Jordan
   canonical forms. Fully justify.

9. (a) What does it mean for a field to be perfect?
   (b) Give an example of a perfect field. (No need to justify your answer.)
   (c) Give an example of a field that is not perfect. (No need to justify your answer.)

10. (a) Classify the conjugacy classes of the symmetric group $S_3$ and justify.
    (b) Construct the character table of $S_3$. 