Algebra Comprehensive Exam

June, 2015

NAME	
SIGNATURE	

- This is a closed-book test. You have 2 hours and 30 minutes to complete the exam.
- The test contains 10 problems. Each problem is worth 10 points.
- **Show all details and quote any theorem you use. We prefer complete solutions of a few problems to many partial solutions.**
- *Please, write clearly and legibly.* Clearly indicate scratch work so it won't be graded.

Problem	1	2	3	4	5	6	7	8	9	10
Score										

Problem 1

Prove that every finite group embeds into $GL_n(\mathbb{R})$.

Problem 2

Show that if G is a finite group of odd order, and $N \subset G$ is a normal subgroup of order 5, then N is in the center of G.

Problem 3

Let A, B, C be finitely generated abelian groups such that $A \oplus C \simeq B \oplus C$. Show that $A \simeq B$.

Problem 4

Show that, if R is an integral domain, then a polynomial of degree d in $R[x]$ can have at most d roots. Give an example of a ring R and a polynomial $f \in R[x]$ of degree d that has more than d roots.

Problem 5

Given an example of a ring and an ideal in this ring which is (a) not principal; (b) prime but not maximal.

Problem 6

If W_1 and W_2 are linear subspaces of a vector space V. Let $\dim W_1 = r$, $\dim W_2 = s$ and $\dim V = n$. What are the possible values of $\dim W_1 \cap W_2$? Prove your answers.

Problem 7

Let A be an $n \times n$ complex matrix such that $A^k = I$ for some k. Show that $|tr(A)| \leq n$.

Problem 8

Let $T : V \to V$ be a normal operator on a Hermitian vector space V. Show that if $v \in V$ is an eigenvector of V, then v is also an eigenvector of T^*.

Problem 9

Suppose E/F is a field extension of degree $[E : F] = p$, and that p is a prime. Show that for any $a \in E$ one either has $F(a) = F$ or $F(a) = E$.

Problem 10

Construct a field with 32 elements. Prove that your construction produces a field with exactly 32 elements.