Algebra Comprehensive Exam

Spring 2013

Each problem is 10 points. Closed book and closed notes. Please turn off and put in your bag your cell phone and anything that has a screen. Time: 2 hours and 30 minutes. Please explain all your answers.

1. Suppose that G is a finite group of odd order and $N \subset G$ is a normal subgroup of order 5. Show that in fact N is central in G, i.e. commutes with all elements.

2. Let $\sigma \in S_n$ be an $(n-1)$ cycle with $n \geq 3$. Show that if $\tau \in S_n$ commutes with σ then τ is in the cyclic subgroup generated by σ.

3. Show that a group of order 340 has a cyclic subgroup of order 85.

4. Let p be a prime, and let R be the ring of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ pb & a \end{pmatrix}$, where a, b are integers. Prove that R is isomorphic to the ring $\mathbb{Z}[\sqrt{p}]$.

5. Let R be a PID and M a free R-module with basis $e_1, \ldots, e_n$. Consider its submodule N generated by $x_1e_1 + \ldots x_ne_n$ for some choice of $x_i \in R$. Suppose that $M = N \oplus L$ for some submodule $L \subset M$. Show that $(x_1, \ldots, x_n) = R$.

6. Let L be a vector space of dimension m and L_1, L_2 its subspaces of dimension m_1, m_2, respectively. Show that $m_1 + m_2 = \dim(L_1 \cap L_2) + \dim(L_1 + L_2)$ and find the range of the possible values for $\dim(L_1 \cap L_2)$ and $\dim(L_1 + L_2)$.

7. Suppose that A is a real matrix of size $n \times n$ and $e^A = 1 + A + A^2/2! + A^3/3! + \ldots$ is its exponent. Show that $\det e^A = 1$ if and only if $tr(A) = 0$. (If you use a formula relating trace and determinant, you need to prove it.)

8. Let S, T be two normal operators on a finite dimensional Hermitian vector space. Show that if $ST = TS$ then ST is normal as well.

9. Show that the polynomials
$$f(x) = x^4 - 2 \qquad \text{and} \qquad g(x) = x^4 + 2$$
have the same splitting field over $\mathbb{Q}$. (Denote this splitting field by K.) Find the degree of the extension $K : \mathbb{Q}$, and a basis of K over $\mathbb{Q}$.

10. For which primes p is the quotient $\mathbb{F}_p[x]/(x^2 + x + 1)$ a field?