Qualifying Examination, June 15, 2016
1:00 pm — 3:30 pm, Room PSCB 120

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Notation:

$\mathbb{C}$ denotes the complex plane; $i = \sqrt{-1}$;
$D(z_0, r)$ denotes the open disc in $\mathbb{C}$ centered at $z_0$ and radius $r$;
$U = \{z = x + iy : y > 0\}$ is the upper half plane in $\mathbb{C}$.
1. Show that

\[ \sum_{n=1}^{\infty} \frac{1}{z^2 + n^2} \]

defines a meromorphic function on \( \mathbb{C} \).
2. Show that for a positive integer \( n \geq 1 \)

\[
\int_0^\infty \frac{1}{x^{2n} + 1} \, dx = \frac{\pi}{2n \sin \frac{\pi}{2n}}.
\]
3. For any non-integers \( \alpha, \beta \) and \( \gamma \), find the radius of convergence for the power series

\[
\sum_{n=0}^{\infty} \frac{\alpha (\alpha + 1) \ldots (\alpha + n - 1) \beta (\beta + 1) \ldots (\beta + n - 1)}{n! \gamma (\gamma + 1) \ldots (\gamma + n - 1)} z^n.
\]
4. Let $f$ be an entire function. Prove the following two statements.
(a) If $|f(z)| \leq M(1 + |z|^n)$ on $\mathbb{C}$ for some positive constant $M$ then $f$ is a polynomial of degree at most $n$.

(b) If $\lim_{|z| \to \infty} |f(z)| = \infty$ then $f$ is a polynomial.
5. Find all entire holomorphic functions $f$ with justification such that

$$\operatorname{Im} f(z) = (y^2 - x^2),$$

where $\operatorname{Im} f$ denotes the imaginary part of $f$. 
6. Prove or disprove: there exists a family \( \{f_n\} \) of holomorphic functions on \( D(0, 2) \) such that \( f_n \to z^3 \) uniformly on the compact set \( \{z \in \mathbb{C} : |z| = 1 \text{ or } 1/2\} \) (two circles: \(|z| = 1\) and \(|z| = 1/2\)).
7. Construct a conformal map \( \phi \) which maps \( D_1 \) onto \( D_2 \), where

\[
D_1 = \{ z = x + iy \in D(0, 1) : y > x \}; \quad \text{and} \quad D_2 = \{ z \in \mathbb{C} : |z| > 1 \}.
\]
8. Let $f : U \to U$ be holomorphic with $U$ being the upper half plane. Prove that

$$|f'(i)| \leq |f(i)|$$

and provide an example indicates the above inequality is an equality.