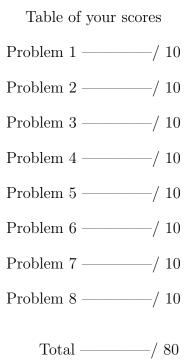
Print Your Name: —	last	first
Print Your I.D. Number:		

Qualifying Examination, June 15, 2016 1:00 pm - 3:30 pm, Room PSCB 120



Notation:

C denotes the complex plane; $i = \sqrt{-1}$; $D(z_0, r)$ denotes the open disc in **C** centered at z_0 and radius r; $U = \{z = x + iy : y > 0\}$ is the upper half plane in **C**. 1. Show that

$$\sum_{n=1}^{\infty} \frac{1}{z^2 + n^2}$$

defines a meromorphic function on ${\bf C}.$

2. Show that for a positive integer $n \ge 1$

$$\int_0^\infty \frac{1}{x^{2n} + 1} \, \mathrm{dx} = \frac{\pi}{2n \sin \frac{\pi}{2n}}.$$

3. For any non-integers α,β and $\gamma,$ find the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{\alpha \left(\alpha+1\right) \dots \left(\alpha+n-1\right) \beta \left(\beta+1\right) \dots \left(\beta+n-1\right)}{n! \gamma \left(\gamma+1\right) \dots \left(\gamma+n-1\right)} z^{n}.$$

4. Let f be an entire function. Prove the following two statements. (a) If $|f(z)| \leq M(1+|z|^n)$ on **C** for some positive constant M then f is a polynomial of degree at most n.

(b) If $\lim_{|z|\to\infty} |f(z)| = \infty$ then f is a polynomial.

5. Find all entire holomorphic functions f with justification such that

$$\operatorname{Im} f(z) = (y^2 - x^2),$$

where ${\rm Im}f$ denotes the imaginary part of f.

6. Prove or disprove: there exists a family $\{f_n\}$ of holomorphic functions on D(0,2) such that $f_n \to \bar{z}^3$ uniformly on the compact set $\{z \in \mathbf{C} : |z| = 1 \text{ or } 1/2\}$ (two circles: |z| = 1 and |z| = 1/2). 7. Construct a conformal map ϕ which maps D_1 onto D_2 , where

$$D_1 = \{z = x + iy \in D(0, 1) : y > x\}; \text{ and } D_2 = \{z \in \mathbf{C} : |z| > 1\}.$$

8. Let $f:U\to U$ be holomorphic with U being the upper half plane. Prove that

$$|f'(i)| \le |f(i)|$$

and provide an example indicates the above inequality is an equality.