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Notation: $D(z_0, r)$ denotes the open disc in $\mathbb{C}$ centered at $z_0$ and radius $r$.

1. Let $f$ be an entire holomorphic function such that $f(z) \not\in \mathbb{R}$ for all $z \in \mathbb{C}$, where $\mathbb{R}$ is the real line in the complex plane $\mathbb{C}$. Prove or disprove $f$ is a constant.
2. Evaluate the real integral

\[ \int_0^\infty \frac{\ln x}{1 + x^4} \, dx \]
3. Let $f$ be entire holomorphic such that $f(x + ix) \in \mathbb{R}$ for all $x \in \mathbb{R}$. If $f(2) = 1 - i$ then find $f(2i)$, where $i^2 = -1$. 
4. Let $h(x)$ be a twice differentiable function on $[-1, 1]$ such that $h(0) = h'(0) = 0$ and $h''(0) \neq 0$. Prove

$$\sum_{n=1}^{\infty} h\left(\frac{1}{n}\right)z^n$$

defines a holomorphic function on $D(0, 1)$ which is continuous on $\overline{D(0, 1)}$. 
5. Let $D$ be a bounded domain in $\mathbb{C}$ with piecewise $C^1$ boundary. Let $f(z)$ be holomorphic in a bounded domain $D$ and $f \in C(\overline{D})$ with all zeros \{ $z_1, \cdots, z_n$ \} $\subset D$ counting multiplicity. Let $g$ be holomorphic in $D$ and continuous on $\overline{D}$. Evaluate

$$\int_{\partial D} \frac{f'(z)}{f(z)} g(z) dz$$
6. Let \( D = \{ z \in \mathbb{C} : |z| < 1, \Re z > 0, \Im z > 0 \} \). Construct a conformal holomorphic map which maps \( D \) onto the unit disc \( D(0,1) \).
7. Let $D$ be a simply connected domain in $\mathbb{C}$ and $z_0 \in D$. Let $\mathcal{F}$ be the set of all $f : D \to D(0,1)$ such that (i) $f(z_0) = 0$; (ii) $f'(z_0) > 0$ and (iii) $f$ is one to one. Then prove $\mathcal{F}$ is not empty set.
8. Let $u(z)$ be harmonic in $D = D(0, 1) \setminus \{0\}$ such that

$$\lim_{z \to 0} \frac{u(z)}{\log |z|} = 0$$

Prove that $u$ can be extended to be harmonic in $D(0, 1)$. 