Print Your Math Exam Id: ______________

Complex Qualifying Examination
Time: 1:00 pm–3:30 pm, 6/21/2017
Room: Rowland Hall, RH114

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Notation:
\( \mathbb{C} \) denotes the complex plane; \( i = \sqrt{-1} \);
\( D(z_0, r) \) denotes the open disc in \( \mathbb{C} \) centered at \( z_0 \) and radius \( r \).
1. Find the integral

$$\int_{0}^{2\pi} \frac{d\theta}{a + \cos \theta}, \quad a > 1.$$
2. The Bernoulli polynomials $B_n(z)$ are defined by the expansion

$$
e^{tz} - 1 \over e^t - 1 = \sum_{n=1}^{\infty} B_n(z) {t^n \over n!}.$$

Prove that $B_n(z + 1) - B_n(z) = nz^{n-1}$. 
3. Let $f(z)$ be analytic in $S = \{ z = x + i y : -1 < x < 1 \}$ and continuous on $\overline{S}$, the closure of $S$. Suppose that $f(z)$ are real when $\text{Re} z = x = \pm 1$. Prove that $f(z)$ can be extended analytically to the whole plane and that the resulting entire function satisfies $f(z + 4) = f(z)$ for all $z \in \mathbb{C}$. 
4. Let $f_n : D(0,1) \to D(0,1) \setminus \{0\}$ be analytic such that $\sum_{n=1}^{\infty} |f_n(0)| < \infty$.

(a) Prove $\sum_{n=1}^{\infty} |f_n(z)|^3$ converges uniformly on $|z| \leq \frac{1}{2}$;

(b) Give an example of $\{f_n\}_{n=1}^{\infty}$ satisfying above conditions but $\sum_{n=1}^{\infty} |f_n(z)|^3$ diverges for any $|z| > 1/2$. 

5. Let $f$ be holomorphic in $D = \{ z \in \mathbb{C} : 2 < |z| < \infty \}$ satisfying

$$\int_{|z|=3} f(z) \, dz = 0.$$ 

Prove that there is a holomorphic function $F$ in $D$ such that $F'(z) = f(z)$ on $D$. 


6. Find a conformal map which maps $U_1$ onto $U_2$, where

$$U_1 = \{ z = x+iy \in \mathbb{C} : y > 0 \} \setminus \{ z = iy : 1 \leq y \leq 2 \} \quad \text{and} \quad U_2 = D(0,1) \setminus \{0\}.$$
7. Let $f$ be meromorphic in $\mathbb{C}$ satisfying

$$|f(z)|^3 \leq |\tan z|, \quad z \in \mathbb{C} \setminus P(f),$$

where $P(f)$ is the set of poles of $f$ in $\mathbb{C}$. Prove $f(z) \equiv 0.$
8. Prove or disprove there is a non-constant entire function \( f = u + iv \) satisfying \( v(z) \neq u(z)^2 \) when \( u(z) \geq 0 \).