Print Your Math Exam Id: ______________

Complex Qualifying Examination

Time: 1:00 pm–3:30 pm, 6/21/2017
Room: Rowland Hall, RH114

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Notation:

$\mathbf{C}$ denotes the complex plane; $i = \sqrt{-1}$;
$D(z_0, r)$ denotes the open disc in $\mathbf{C}$ centered at z_0 and radius r.

1. Find the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \cos\theta}, \quad a > 1.$$

2. The Bernoulli polynomials $B_n(z)$ are defined by the expansion

$$t\frac{\mathrm{e}^{\mathrm{tz}}-1}{\mathrm{e}^{\mathrm{t}}-1}=\sum_{n=1}^{\infty}\frac{B_n\left(z\right)}{n!}t^n.$$

Prove that $B_n\left(z+1\right)-B_n\left(z\right)=nz^{n-1}$.

3. Let $f(z)$ be analytic in $S = \{z = x + iy : -1 < x < 1\}$ and continuous on $\overline{S}$, the closure of S. Suppose that $f(z)$ are real when Re $z = x = \pm 1$. Prove that $f(z)$ can be extended analytically to the whole plane and that the resulting entire function satisfies $f(z + 4) = f(z)$ for all $z \in \mathbf{C}$.

4. Let $f_n : D(0,1) \to D(0,1) \setminus \{0\}$ be analytic such that $\sum_{n=1}^{\infty} |f_n(0)| < \infty$.

(a) Prove $\sum_{n=1}^{\infty} |f_n(z)|^3$ converges uniformly on $|z| \leq \frac{1}{2}$;

(b) Give an example of $\{f_n\}_{n=1}^{\infty}$ satisfying above conditions but $\sum_{n=1}^{\infty} |f_n(z)|^3$ diverges for any $|z| > 1/2$.

5. Let f be holomorphic in $D = \{z \in \mathbf{C} : 2 < |z| < \infty\}$ satisfying

$$\int_{|z|=3} f(z)dz = 0.$$

Prove that there is a holomorphic function F in D such that $F'(z) = f(z)$ on D.

6. Find a conformal map which maps U_1 onto U_2, where

$U_1 = \{z = x+iy \in \mathbf{C} : y > 0\} \setminus \{z = iy : 1 \leq y \leq 2\}$ and $U_2 = D(0,1) \setminus \{0\}$.

7. Let f be meromorphic in $\mathbf{C}$ satisfying

$$|f(z)|^3 \leq |\tan z|, \quad z \in \mathbf{C} \setminus P(f),$$

where $P(f)$ is the set of poles of f in $\mathbf{C}$. Prove $f(z) \equiv 0$.

8. Prove or disprove there is a non-constant entire function $f = u + iv$ satisfying $v(z) \neq u(z)^2$ when $u(z) \geq 0$.