Print Your Name: last first

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Comprehensive Examination of Analysis
9:00AM–11:30AM, June 16, 2015; Rowland Hall 114

Choose 8 from the 9 problems
You need to cross out the problem you don’t want to be graded

Problem 1 ———— / 10
Problem 2 ———— / 10
Problem 3 ———— / 10
Problem 4 ———— / 10
Problem 5 ———— / 10
Problem 6 ———— / 10
Problem 7 ———— / 10
Problem 8 ———— / 10
Problem 9 ———— / 10

Total ———— / 80
1. Prove that

\[ \sum_{n \geq 2} \frac{1}{n \log^2 n} < +\infty. \]
2. Compute

\[ \lim_{n \to +\infty} \int_0^1 \sin(nx) e^{-x^2} \, dx. \]

Justify your answer.
3. Assume that $f \in C^1(\mathbb{R})$ and $\lim_{|x| \to +\infty} \frac{f(x)}{|x|} = +\infty$. Show that for any $p \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $f'(y) = p$ (i.e., $f' : \mathbb{R} \to \mathbb{R}$ is onto).

Hint: Consider $g(x) = f(x) - px$ and $\lim_{x \to \pm\infty} g(x)$. 


4. (a) State the Stokes’ Theorem
(b) Evaluate the following integral:

\[ \int_{\partial D} \frac{x^3}{3} dy \wedge dz + \sin(yz) dy \wedge dz + x^{10} dx \wedge dz \]

where

\[ D = \{ (x, y, z) : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} < 1 \} \]
5. Prove that \( \sin(\sqrt{x}) \) is uniformly continuous on \([0, \infty)\).
6. Let

\[ f_n(x) = \frac{nx}{1 + n^2x^3}. \]

(a) Prove \( f_n(x) \to 0 \) as \( n \to \infty \) pointwisely on \([0, \infty)\);
(b) Prove or disprove \( f_n(x) \to 0 \) as \( n \to \infty \) uniformly on \([0, \infty)\).
7. Let \((X, d)\) be a metric space and \(E \subset X\) is a compact set. Prove that \(E\) is closed.
8. Let \( f(x, y) \) be a function on the unit disc \( D = \{(x, y) : x^2 + y^2 < 1\} \) with \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) exist for all \((x, y) \in D\). Prove or disprove the following each statement.

   (a) If \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) are bounded on \( D \), then \( f \) is continuous on \( D \);

   (b) If \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) are continuous on \( D \), then \( f \) is differentiable on \( D \).
9. Let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Prove

(a) For any $x, y \in (0, \infty)$

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

(b) If $f$ and $g$ are in $L^p[a, b]$ and $L^q[a, b]$, respectively, then $f(x)g(x)$ is Lebesgue integrable and we have

$$\int_a^b f(x)g(x)dx \leq \left( \int_a^b |f(x)|^p dx \right)^{1/p} \left( \int_a^b |g(x)|^q dx \right)^{1/q}.$$
Problem 9 (continued)