Print Your Name: ______________ ______________
last first

Print Your I.D. Number: __________ ______________

Complex Qualifying Examination
Time: 1:00 pm–3:30pm, September 20, 2012
Room: Rowland Hall 114

Choose any 8 problems from 9

Table of your scores

Problem 1 ________/ 10

Problem 2 ________/ 10

Problem 3 ________/ 10

Problem 4 ________/ 10

Problem 5 ________/ 10

Problem 6 ________/ 10

Problem 7 ________/ 10

Problem 8 ________/ 10

Problem 9 ________/ 10

Total ________/ 80

Notation. Let $D(z_0, R)$ denote the disc in $\mathbf{C}$ centered at z_0 with radius R.

1. Show that for $a > 0$,

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2} dx = \frac{\pi(1+a)}{4e^a}.$$

2. Suppose f is analytic in an annulus $A(0, r, R) = \{z \in \mathbf{C} : r < |z| < R\}$, and there exists a sequence of polynomials p_n converging to f uniformly on compact subsets of $A(0, r, R)$. Show that f is an analytic function on the disc $D(0, R)$.

3. Let $P(z)$ be a polynomial in z. Assume that $P(z) \neq 0$ for $\mathrm{Re}(z) > 0$. Show that $P'(z) \neq 0$ for $\mathrm{Re}(z) > 0$.

4. Determine the number of roots, counted with multiplicity, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $A(0, 1, 2) = \{z \in \mathbf{C} : 1 < |z| < 2\}$.

5. Let f be analytic on the upper-half plane and satisfy $|f(z)| < 1$. Furthermore suppose $f(i) = 0$. Give an upper bound for $|f'(i)|$ and state which functions realize this extrema.

6. Let $f(z)$ be an entire holomorphic function on $\mathbf{C}$ such that

$$|f(e^z)| \leq |e^z|, \quad z \in \mathbf{C}$$

Prove $f(z) \equiv az$ for some constant $|a| \leq 1$.

7. Let f be holomorphic in $D(0,1) \setminus \{0\} = \{z \in \mathbf{C} : 0 < |z| < 1\}$. If

$$\int_D |f(z)|^3 dA(z) = \int_D |f(x+iy)|^3 dx dy < \infty$$

then $z = 0$ is removable singularity of f.

8. Find the largest set in $\mathbf{C}$ where the Laurent series

$$\sum_{j=-\infty}^{\infty} 2^j z^{j^3}$$

converges.

9. Let u be a real-valued harmonic function in $\mathbf{C}\backslash\{0\}$. Show that then

$$u(z) = c \log |z| + \operatorname{Re}(f(z))$$

for some real constant c and a holomorphic function f on $\mathbf{C}\backslash\{0\}$.